Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Colouring sets of cubic 6-poles

Bachelor Thesis

2023
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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Robert Lukoťka, PhD.

Bratislava, 2023
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79158997

Univerzita Komenského v Bratislave
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Študijný odbor:	informatika
Typ záverečnej práce:	bakalárska
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: Colouring sets of cubic 6-poles
Farebné množiny kubických 6-pólov

Anotácia: Kubický k-pól P je graf, ktorý má k usporiadaných vrcholov stupňa jedna a ostatné vrcholy majú stupeň tri. Keď zafarbíme P farbami {0, 1, 2}, hraničné farbenie je k-tica, ktorej i-tým prvkom je farba hrany incidentná s i-tým vrcholom stupňa jeden. Farbiaca množina P je množina všetkých hraničných farbení P. Známe sú dve nutné podmienky, ktoré musí farbiaca množina spĺňať: paritná lema a silná Kempe uzavretosť. Prvky {0, 1, 2}^k spĺňajúce tieto podmienky nazývame prípustné farbiace množiny. Na množine prípustných farbiacich množín možno definovať reláciu ekvivalencie: Dve prípustné farbiace množiny sú ekvivalentné, ak majú rovnakú množinu
prípustných farbiacich množín vo svojom komplemente. Túto reláciu ekvivalencie možno skombinovať s ďalšími dvoma reláciami ekvivalencie: preusporiadanie vrcholov stupňa jedna a zámenou farieb. Takto možno vytvoriť zjednocujúcu reláciu ekvivalencie. Chceme študovať otázku, ktoré triedy tejto ekvivalencie obsahujú farbiacu množinu nejakého 6-pólu.

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Spôsob sprístupnenia elektronickej verzie práce:
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Dátum zadania: 06.10.2022

Dátum schválenia: 06.10.2022

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THESIS ASSIGNMENT

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Field of Study:	Computer Science
Type of Thesis:	Bachelor´s thesis
Language of Thesis:	English
Secondary language:	Slovak

Title: Colouring sets of cubic 6-poles

Annotation: Cubic k-pole P is a graph that has k ordered vertices of degree one, and the remaining vertices are of degree three. Given a proper colouring of P with colours {0, 1, 2}, the boundary colouring is the k-tuple whose i-th element is the colour of the edge incident to the i-th vertex of degree
one. A colouring set of P is the set of possible boundary colourings of P. There are two obvious necessary conditions that state when an element from {0, 1, 2}^k may be a colouring set of a k-pole, one is given by parity lemma, the second is given by the fact that the colouring set needs to be strongly Kempe closed. We call such colouring sets plausible. An equivalence relation on plausible colouring sets can be defined as follows: two colouring sets are equivalent if they have the same set of plausible colouring sets in the complement. This relation behaves nicely with respect to two other equivalence relations obtained by reordering vertices of degree one and colour swapping, which gives rise to one unifying equivalence relation. We want to study which equivalence classes of the unifying equivalence contain a colouring set of some cubic 6-pole.

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Acknowledgments: I want to thank my supervisor for the invaluable guidance and countless pieces of advice he provided throughout this thesis.

Abstrakt

Dôležitou súčasťou tejto práce sú snarky, kubické grafy bez hranového 3-farbenia. Bolo vykonaných viacero prác, ktoré skúmali dekompozície a redukcie snarkov. K-redukciám snarkov už rozumieme úplne pre ľubovoľné k [Nedela, Škoviera, Decompositions and Reductions of Snarks, 1996]. Analýza k-dekompozícií je však kompletná len pre $k \leq 5$. Táto práca nadväzuje na skúmanie 6-dekompozícií v práci [Karabáš, Máčajová, Nedela, 6-decompositions of snarks, 2013].

K-dekompozícia snarku je rozdelenie snarku k-hranovým rezom a pridanie malého počtu vrcholov a hrán k výsledným komponentom, tak aby nám vznikli snarky menšieho alebo rovnakého stupňa.

Aby sme lepšie pochopili k-dekompozíciu, skúmali sme regulárne hranové 3-farbenie kubických k-pólov, čo sú grafy so všetkými vrcholmi stupňa tri okrem k usporiadaných vrcholov stupňa jeden. Farebná množina je množina k-tíc farieb 0, 1 a 2, ktoré sa inak nazývajú aj hraničné farbenia. Pri ľubovoľnom regulárnom hranovom 3-farbení kubického k-pólu je jeho hraničné farbenie n-tica farieb jeho hrán incidentných s vrcholmi stupňa jeden. Farebná množina kubického k-pólu je množina všetkých jeho možných hraničných farbení.

Hlavným cieľom tejto práce bolo identifikovať realizovateľné farebné množiny, definované ako tie, ktoré sú farebné množiny niektorého kubického k-pólu. Farebné množiny, ktoré spĺňajú dve nevyhnutné podmienky, Paritnú Lemu a Kempe uzavretosť, budeme označovať ako prípustné. Dve relácie ekvivalencie možno zjednotiť do sk-ekvivalencie: jedna, v ktorej sú množiny farieb s rovnakým príspustným doplnkom ekvivalentné, a druhá, ktorú môžeme získať permutáciou hrán.

Zjednocujúca sk-ekvivalencia rozdeľuje množiny farieb do 170 tried. Pre účely k-dekompozície sme potrebovali nájsť len jednu množinu farieb z každej triedy. Sk-ekvivalencia nám tiež umožňuje efektívnejšie realizovať takéto farebné množiny. Na určenie realizovateľnosti farebných množín sme aplikovali sériu algoritmov, od rozširovania farebnej množiny prázdneho grafu pridaním hrán až po kombinovanie existujúcich farebných množín realizovateľných k-pólov. Táto práca úspešne identifikuje 115 z 170 tried farebných množín. Zostávajúcich 55 tried predstavuje približne 0,1% všetkých prípustných farebných množín, čo naznačuje ich zriedkavosť a potenciálnu nerealizovateľnosť.

Kľúčové slová: snark, kubický k-pól, regulárne hranové 3-farbenie, k-dekompozícia

Abstract

An important part of this thesis are snarks, cubic graphs with no 3-edge-colouring. There have been several works which investigated the decompositions and reductions of snarks. The complete understanding of k-reductions of snarks, for any given k, was already achieved [Nedela, Škoviera, Decompositions and Reductions of Snarks, 1996]. However, the analysis of k-decompositions is only complete for $k \leq 5$. This work builds upon the exploration of 6-decompositions in work [Karabáš, Máčajová, Nedela, 6-decompositions of snarks, 2013].

K-decomposition of a snark is dividing a snark with a k-edge-cut and adding a small number of vertices and edges to resulting components to make them snarks of a smaller or equal order.

To better understand k-decomposition, we were studying the proper 3-edge-colourability of cubic k-poles, which are graphs with all vertices of degree three except for k ordered vertices of degree one. A colouring set is a set of k-tuples of colours 0, 1, and 2, called boundary colourings. Given any proper 3-edge-colouring of a cubic k-pole, its boundary colouring is the tuple of colours of its edges incident to vertices of degree one. The colouring set of a cubic k-pole is a set of all its possible boundary colourings.

The primary objective of this thesis was to identify realisable colouring sets, defined as those that are colouring sets of some cubic k-pole. We will refer to colouring sets that satisfy two necessary conditions, the Parity Lemma and Kempe Closeness, as plausible. Two equivalence relations can be unified into sk-equivalence: one under which colouring sets with the same plausible complement are equivalent, and the second, which can be obtained by permutating edges.

The unifying sk-equivalence divides the colouring sets into 170 classes. For the purposes of k-decomposition, we only needed to find one colouring set from each class. Sk-equivalence also helps us realise such colouring sets more effectively. A range of algorithms is applied to determine the realisability of the colouring sets, from the expansion of a colouring set of an empty graph by adding edges to combining existing colouring sets of realised k-poles. This study successfully identifies 115 out of 170 colouring set classes. The remaining 55 classes constitute roughly 0.1% of all plausible colouring sets, suggesting their rarity and potential non-realisability.

Keywords: snark, cubic k-pole, proper 3-edge-colouring, k-decomposition

Contents

List of Figures

List of Tables

Introduction

Cubic graphs and especially snarks are an important part of graph theory. A *snark* is a cubic graph that does not have a proper 3-edge-colouring.

Many graph theory problems can be reduced to a problem involving the 3-edge-colourability of cubic graphs. For example, the Four colour theorem, which states that the chromatic number of every loopless planar graph is at most 4 is equivalent to a problem that every snark is non-planar, as was proven by Peter G. Tait in 1880 [1]. Another example indicating the importance of snarks is the Cycle double cover conjecture which asks whether every bridgeless graph has a multiset of cycles covering every edge exactly twice. Snarks are the hardest part of this problem, i.e. if the conjecture holds true for snarks, it must be true for every graph [2].

A *k-edge-cut* of a connected graph is a set of edges, whose removal disconnects the graph. Informally speaking, a *k-decomposition* of a snark involves dividing the snark into two components using a k-edge-cut and adding a small number of vertices and edges to complete the components into snarks of a smaller or same order.

One way to interpret the k-decomposition of a snark is to split the snark into two cubic k-poles, X and Y. A *cubic k-pole* is a graph with k ordered vertices of degree 1, whose incident edges are sometimes referred to as *dangling edges*, and the remaining vertices of degree 3. Once we have the two cubic k-poles X and Y, we connect them to new cubic k-poles, X_2 and Y_2, respectively. This is done by connecting the i-th dangling edge of the first k-pole and the i-th dangling edge of the second k-pole, effectively turning them into a single edge.

For small k, a k-decomposition of any snark can be performed by adding only a few new vertices to the components. Specifically, for $k = 2, 3, 4$, and 5, adding $0, 1, 2$, and 5 new vertices, respectively, is sufficient, as demonstrated by Goldberg for $k = 4$ [3] and Cameron et al. for $k = 5$ [4]. However, the cases where k is greater than or equal to 6 are still unresolved.

Boundary colourings are k-tuples of colours 0, 1, 2. Given any proper 3-edge-colouring of a cubic k-pole, the k-tuple of colours of its dangling edges is its *boundary colouring*. We will refer to any set of boundary colourings of the same length as a *colouring set*. A *colouring set of a cubic k-pole* consists of all its boundary colourings. A colouring set is *realisable* if and only if it is a colouring set of some cubic k-pole.

Clearly, studying colouring sets will help us to determine whether and how a cubic k-pole can be completed into a snark.

In this bachelor's thesis, we mainly focused on figuring out which colouring sets of cubic 6-poles are realisable. We used several ways to reduce the number of colouring sets we needed to check.

A *c-equivalence* is an equivalence relation on boundary colourings, such that two boundary colourings are c-equivalent, if and only if there is a colour mapping that transforms the first boundary colouring into the second one (for example $(0, 0, 1, 1, 2, 2) \equiv (1, 1, 2, 2, 0, 0)$). Because we can always recolour the whole graph with this mapping, we can see that for a given c-equivalence class, either all elements need to be in the colouring set of a k-pole or none of them. This condition is therefore necessary for a colouring set to be realisable. Possible colouring sets are then described solely by which c-equivalence classes they contain. Other known necessary conditions for the realisability of a colouring set are the Parity lemma and Kempe-closeness (colouring sets satisfying all three necessary conditions are called *plausible*). Similarly, for a given colouring set we can change the order of dangling edges to get a new colouring set. We will call this change of order *s-equivalence*.

Since we are primarily interested in colouring sets' complements to a snark we also introduce a new equivalence (*k-equivalence*) which divides colouring sets into classes based on what colouring sets they can connect with to create a snark. As we will demonstrate later in Chapter 2, this equivalence can help us search for new realisations of colouring sets more effectively.

By determining the representatives of s-equivalence, k-equivalence and unifying sk-equivalence (as described in Chapter 3), we reduced the total number of colouring sets we needed to realise to 170. We used several algorithms to realise them. Our first algorithm expands a colouring set of an empty graph by adding one edge at a time in various ways while tracking the corresponding colouring set. Other algorithms are mainly focused on combining already realised colouring sets of k-poles (for $k = 4, 5, 6, 7$) while filtering some k-poles and focusing on combinations that have a higher chance of success.

This way we managed to find 115 out of 170 colouring set classes. Colouring sets from the remaining 55 classes represent approximately 0.1% of all plausible colouring sets (without taking sk-equivalence into account), so they are very rare and it is possible some of them cannot be realised.

Chapter 1

Preliminaries

First, we introduce some concepts that are needed to understand the topic.

Definition 1. *A graph is a pair (V, E), where V is a set of vertices and E is a multiset of elements, known as edges, from* $\{\{v_1, v_2\} \mid v_1, v_2 \in V\}$.

This definition is, therefore, similar to the commonly used definition of a graph, but ours also allows loops (edges that connect a vertex to itself) and multiple edges between the same pair of vertices. Additionally, in our graph model, a loop is considered to be incident with itself. The degree of a vertex is twice the number of its loops plus the number of other incident edges.

Definition 2. *A proper k-edge-colouring of a graph G = (V, E) is a map* $c : E \to \{0, \dots, k-1\}$ *where* $c(e) \neq c(f)$ *for all pairs of adjacent edges e, f.*

Since our primary concern is proper 3-edge-colouring, we will simply refer to this as *colouring*. The graph with loops is therefore not colourable.

A *cubic k-pole* is a graph with all vertices of degree three except k ordered vertices of degree one. We will call edges incident with vertices of degree one *dangling edges*. We may also call a graph a *cubic pole* if we do not want to specify the number of dangling edges. A *boundary colouring of degree k* is an element from $\{0, 1, 2\}^k$. If a cubic pole C can be coloured so that its dangling edges are coloured with a boundary colouring B, we will say that B is a boundary colouring of C. The set of some boundary colourings of degree k is a *colouring set of degree k*. A *colouring set* of a cubic k-pole C is the set of all its boundary colourings. A colouring set is *realisable* if and only if it is a colouring set of some cubic k-pole C.

Lemma 1. *(Parity Lemma). Let P be a cubic k-pole with a colouring and let* k_1, k_2 *and* k_3 *be the numbers of occurrences of colours in a boundary colouring. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

Specifically, for k = 6, the number of occurrences of each colour is even.

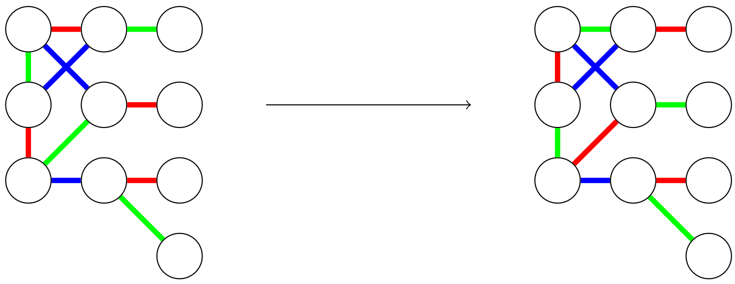

Figure 1.1: Example of a Kempe switch

Proof. Without loss of generality, we will prove the lemma only for colour 0. Let us denote V_i as the vertices of degree i, ND as the set of non-dangling edges coloured 0, and D as the set of dangling edges coloured 0. Our aim is to show that the parity of $|V_1|$ and $|D|$ is the same.

Since a cubic k-pole has all vertices of odd degree, it must have an even number of vertices. This implies that the parity of $|V_3|$ and $|V_1|$ is the same. Edges from ND connect two vertices from V_3, while edges from D connect one vertex from V_3 and one vertex from V_1.

As each vertex is connected to exactly one edge coloured 0, we have: $|V_3| = 2 \cdot |ND| + |D|$, which means $|V_3|$ and $|D|$ have the same parity. This implies that the parity of $|V_1|$ and $|D|$ is also the same. □

It is evident that all realisable colouring sets only contain boundary colourings that satisfy Parity Lemma.

Let us define an *index pairing* as a set of index pairs $\{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\}$ where each index from the set $\{0, 1, \ldots, k-1\}$ appears at most once. An index pairing of colours $\{a, b\}$ of a boundary colouring is an index pairing such that only indices with colours a or b are used and they are each used exactly once. For instance, if a boundary colouring is represented by (0, 0, 1, 2, 2, 1), it can have an index pairing of colours $\{0, 1\}$ such as $\{(0, 5), (1, 2)\}$.

A *Kempe chain* is a maximal path with edges of alternating colours $\{a, b\}$. In the case of a cubic pole, we can see that a Kempe chain starts and ends with a dangling edge. A *Kempe switch* is a transformation of the colouring of a cubic pole such that edges from a Kempe chain (described above) coloured a are changed to colour b and vice versa.

We can see that for any given cubic pole and its colouring, there is exactly one index pairing of colours $\{a, b\}$ that describes possible kempe switches. Each Kempe switch swaps two colours of the boundary colouring according to one pair of indices. This gives us another necessary property for a colouring set to be realisable.

Definition 3. *A colouring set C is called Kempe closed if for every boundary colouring B from C and every set of colours $\{a,\ b\}$ there exists an index pairing of $\{a,\ b\}$ of B*

called P such that for all index pairings that are a subset of P, by transforming the boundary colouring with the corresponding Kempe switches we will obtain a boundary colouring present in C.

If a boundary colouring B from a colouring set C satisfies the condition above, we will say that B *satisfies Kempe closeness.* A colouring set is therefore considered Kempe closed if all its boundary colourings satisfy Kempe closeness.

1.1 C-equivalence

Considering only 6-poles, the number of all boundary colourings is 3^6. The number of colouring sets is, therefore, 2^{3^6}. This number seems large, but we will show that most of them are redundant. The equivalence described in this chapter has been used before in a work by Karabáš, Máčajová and Nedela [5].

Definition 4. *We consider two boundary colourings* $(x_1, x_2, ..., x_k)$*,* $(y_1, y_2, ..., y_k)$ *c-equivalent if and only if* $(x_1, x_2, ..., x_k) = (\phi(y_1), \phi(y_2), ..., \phi(y_k))$ *for some permutation* ϕ *of colours* $\{0, 1, 2\}$*.*

Lemma 2. *Let us have two boundary colourings* $x = (x_1, x_2, ..., x_k)$ *and* $y = (y_1, y_2, ..., y_k)$ *that are c-equivalent and a cubic pole G. The colouring set of G contains x if and only if it contains y.*

Proof. Let us have two c-equivalent boundary colourings x and y and ϕ such that the above holds. If G contains x, then there exists a colouring, such that boundary edges are coloured by x. Then we can replace all colours m by $\phi(m)$ and we get a new proper colouring with boundary colouring y. Therefore G also contains y. □

Due to this condition, for a cubic pole G to be realisable, its colouring set must include either all or none of the boundary colourings for each c-equivalence class. We will refer to such colouring sets as those that *satisfy c-equivalence.* Hence, we only need to be concerned with one (canonical) boundary colouring for each c-equivalence class. For cubic 6-poles, this reduces the number of boundary colourings we need to consider from 3^6 to just 31 (we are only considering boundary colourings, that satisfy the Parity lemma) and the number of colouring sets we need to consider to 2^{31}. The list of representatives of given c-equivalence classes indexed the same way as in our source code is given in the table 1.1 (representatives for boundary colourings of different degrees can also be found in the source code). We will also call c-equivalence classes *colour types.*

Therefore we only need 31 bits to represent each colouring set. Each bit is set to 1 if our colouring set contains all boundary colourings of a given colour type and 0

Table 1.1: Representatives of c-equivalence classes

Index	Boundary Colouring	Index	Boundary Colouring
0	(0,0,0,0,1,1)	15	(0,0,1,1,2,2)
1	(0,0,0,1,0,1)	16	(0,0,1,2,1,2)
2	(0,0,0,1,1,0)	17	(0,0,1,2,2,1)
3	(0,0,1,0,0,1)	18	(0,1,0,1,2,2)
4	(0,0,1,0,1,0)	19	(0,1,0,2,1,2)
5	(0,0,1,1,0,0)	20	(0,1,0,2,2,1)
6	(0,1,0,0,0,1)	21	(0,1,1,0,2,2)
7	(0,1,0,0,1,0)	22	(0,1,1,2,0,2)
8	(0,1,0,1,0,0)	23	(0,1,1,2,2,0)
9	(0,1,1,0,0,0)	24	(0,1,2,0,1,2)
10	(1,0,0,0,0,1)	25	(0,1,2,0,2,1)
11	(1,0,0,0,1,0)	26	(0,1,2,1,0,2)
12	(1,0,0,1,0,0)	27	(0,1,2,1,2,0)
13	(1,0,1,0,0,0)	28	(0,1,2,2,0,1)
14	(1,1,0,0,0,0)	29	(0,1,2,2,1,0)
		30	(0,0,0,0,0,0)

otherwise. We will call such a sequence of bits a colouring bit array or CBA for short. A degree of CBA is the degree of a colouring set it represents.

A *plausible* colouring set is a set that meets three conditions: it satisfies c-equivalence, is Kempe closed, and satisfies the Parity Lemma.

Lemma 3. *[5] Colouring set of every cubic k-pole is plausible.*

1.2 K-decomposition

Consider a cubic k-pole X and its two dangling edges - (v_1, w_1) and (v_2, w_2). Here, v_1 and v_2 are vertices of degree one. The vertices w_1 and w_2 may represent the same or different vertices. We can form a new cubic $(k-2)$-pole X' by eliminating both edges and the vertices v_1 and v_2, and introducing a single edge (w_1, w_2) in their place. We will say that X' was created from X through a junction of edges (v_1, w_1) and (v_2, w_2). Additionally, if a cubic k-pole Y has an edge (v_1, v_2) connecting two vertices of degrees one and Y' is the graph after the removal of those vertices and an edge, we will say that Y' was created from Y through a junction of edge (v_1, v_2).

Definition 5. *Let Z be a cubic (m+n)-pole with two components X and Y, such that X is a cubic m-pole and Y is a cubic n-pole. Let $(e_1, \ldots, e_k)$ and $(f_1, \ldots, f_k)$ be some* *dangling edges of X and Y respectively. Then Z' is a k-junction of X and Y (denoted as $Z' = X *_k Y$) created by performing a junction of edges (e_i, f_i) for all $1 \leq i \leq k$.*

Definition 6. *Let G be a snark and a k-junction of two 3-edge-colourable cubic k-poles X and Y. If there exist cubic k-poles X' and Y' such that $G_1 = X' *_k X$, $G_2 = Y' *_k Y$ and G_1, G_2 are snarks with orders at most the order of G, then $\{G_1, G_2\}$ is a k-decomposition of G. If G_1, G_2 have orders less then the order of G, we will call such k-decomposition proper.*

We may also not specify k and say that Z is a junction of X and Y $(Z = X * Y)$.

Definition 7. *A set of k edges whose removal disconnects the graph is called k-edge-cut. If each component contains at least one cycle, we also call such a set of k edges cyclic k-edge-cut.*

Definition 8. *Cyclic edge connectivity of a graph is the smallest k, such that the graph has a cyclic k-edge-cut.*

For a given k, it is interesting to study how many new vertices (i. e. order of X' and Y' in Definition 6) need to be added to each component to perform the k-decomposition. For $k = 2, 3, 4, 5$, the problem has already been solved and the number of new vertices needed is 0, 1, 2 and 5 respectively [3] [4]. Jaeger and Swart [6] conjectured that no cyclically 7-connected snark exists. That means it is worth studying k-decomposition for $k \leq 6$. If for a k-pole X, there exists another k-pole Y, that has a smaller number of vertices and its colouring set is a subset of a colouring set of Y, we call Y *reducible*, otherwise, we call it *irreducible*. Irreducible k-pole are interesting because they are strictly better for completing k-poles to snarks than reducible k-poles. Karabáš, Máčajová and Nedela found 14 such 6-poles that can be used to perform a 6-decomposition of all snarks of order 30 or less [5]. They also showed that either this set is enough to perform all 6-decompositions on snarks of arbitrary size or we need 6-poles of size at least 20 to decompose some snarks.

1.3 S-equivalence

Definition 9. *We consider colouring sets X and Y s-equivalent if and only if for some permutation ϕ of $\{1, 2, ..., k\}$:*

$$X = \{(x_1, x_2, ..., x_k) \mid \exists y \in Y : y = (x_{\phi(1)}, x_{\phi(2)}, ..., x_{\phi(k)})\}$$

In other words, X and Y are s-equivalent if and only if one can be obtained from another by permutating all tuples the same way.

Lemma 4. *Let X and Y be s-equivalent sets. X is realisable if and only if Y is realisable.*

Proof. Let us denote the permutation such that the equation in Definition 9 holds as ϕ. If X is realisable, a cubic k-pole G exists with a given colouring set. Its boundary edges are ordered and if we change their order with ϕ we get a new cubic k-pole H. Its colouring set is clearly Y. □

This lemma implies that either all or none of the colouring sets from each s-equivalence class are realisable. From each s-equivalence class of plausible colouring sets, we will choose one representative.

Clearly, it suffices to realise one CBA from each s-equivalence class.

Chapter 2

K-equivalence

This chapter is only partly my work, I was provided lemmas with proofs by my supervisor, and I only made some of them stronger or found simpler proofs for some of them. The way most terms are defined was also mostly done by me to make it the most suitable for this thesis.

Definition 10. *A k-junction of colouring sets M and N is defined as:*

$$\begin{aligned} M *_k N = \{(&m_1, \dots, m_{i_1-1}, m_{i_1+1}, \dots, m_{i_2-1}, m_{i_2+1}, \\ &\dots, m_{i_k-1}, m_{i_k+1}, \dots, m_r, n_1, \dots, n_{i_1-1}, n_{i_1+1}, \\ &\dots, n_{i_k-1}, n_{i_k+1}, \dots, n_t) \mid (m_1, \dots, m_r) \in M, \\ &(n_1, \dots, n_t) \in N, \forall x \in \{1, \dots, k\} : m_{i_x} = n_{i_x}\} \end{aligned}$$

where $i_1, \dots, i_k$ are some indices.

This definition represents what happens to colouring sets if we do a k-junction of two cubic poles.

Definition 11. *Given a plausible colouring set C of degree k, the complements of C are defined as the set of plausible colouring sets X of degree k, such that $C \cap X = \emptyset$.*

In other words, they are plausible colouring sets disjoint with C. If this holds true for colouring sets C_1 and C_2 of cubic k-poles X_1 and X_2, we can see that a junction of X_1 and X_2 is a snark.

Lemma 5. *For every plausible colouring set C, there exists a greatest element in the set of complements of C, with respect to the inclusion relation.*

Proof. We will denote the set of complements of C as D. Let us look at any two plausible sets D_1 and D_2 from D. Because D_1, D_2 are disjoint with C, $D_1 \cup D_2$ is also disjoint with C. Also because D_1 and D_2 satisfy c-equivalence and parity lemma, $D_1 \cup D_2$ does also. If a boundary colouring B in D_1 satisfies Kempe closeness with some index pairing P, we can see that B will still satisfy Kempe closeness with index pairing P for any superset of D_1. That means all boundary colourings from D_1 in $D_1 \cup D_2$ satisfy Kempe closeness. The same can be said for D_2 so $D_1 \cup D_2$ is Kempe closed. This means $D_1 \cup D_2$ is a plausible set disjoint with C so it is also in D. We proved that for any D_1 and D_2 from D, $D_1 \cup D_2$ is also in D, which together with the fact that there are only finitely many boundary colourings implies there is the greatest element in D. □

We will call this greatest element *the reduced complement of C.*

Definition 12. *We consider plausible colouring sets X and Y k-equivalent if and only if their reduced complement is the same.*

Essentially, k-equivalence is dividing colouring sets into equivalence classes based on what colouring sets they are disjoint with. By doing a transitive closure of a union of s-equivalence and k-equivalence, we get *sk-equivalence.*

The motivation behind defining k-equivalence is clear. We need to solve the 6-decomposition for only one representative from each k-equivalence class (i.e. by finding the best way to connect a cubic k-pole to a snark, all cubic k-poles in the same k-equivalence class have the same solution).

We will call the k-equivalence (or sk-equivalence) class *realisable* if at least one of its colouring sets is realisable.

Lemma 6. *Let us have a colouring set X that is a k-junction of two colouring sets A and B. Let us replace A with k-equivalent colouring set A'. Newly created colouring set X' that is a k-junction of A' and B is then k-equivalent to X.*

Proof. We will prove this by contrapositive. In other words, we will prove that X is not k-equivalent to X' implies that A is not k-equivalent to A'. Since X is not k-equivalent to X' there exists a colouring set C, such that $X \cap C = \emptyset$ and $X' \cap C \neq \emptyset$ or vice versa. Now $B * C$ is a colouring set that is disjoint with A but it is not disjoint with A' (or vice versa). Therefore A and A' are not k-equivalent. □

Lemma 7. *Let us have two cubic poles X and X' such that $X = A{*_k}B$ and $X' = A'{*_k}B$. Cubic poles A and B do not need to have the same degree or degree k. Both k-junctions are using the same set of edges from B. If A is k-equivalent to A' then X is k-equivalent to X'.*

Lemma 7 says the same thing as Lemma 6, but for colouring sets. Cubic poles from Lemma 7 have corresponding colouring sets for which Lemma 6 holds, so the Lemma 7 must hold also.

As we will see in Chapter 4, Lemma 7 is very important for realising sk-equivalence classes, because we can work with one representative from each sk-equivalence class instead of all plausible colouring sets. This greatly reduces the amount of work that needs to be done.

Chapter 3

Determining sk-equivalence classes

Our goal is to create a list of plausible CBAs and representatives of s-equivalence, k-equivalence and sk-equivalence in the form of CBAs. We mainly care about colouring sets of degree 6, but lists of CBAs for lower degrees can also be useful.

Because the number of all CBAs of colouring sets of degree 6 is 2^{31} (for lower degrees it is even lower), we can start by generating all of them. The number of CBAs of degree 7 and higher is too big (2^{91}, because we have 91 colour types), so we cannot search through them all effectively.

For each CBA C, we need to determine if it is Kempe closed. This can be done by verifying if each boundary colouring has an index pairing such that the set of boundary colourings obtainable from this index pairing is a subset of C. If this condition holds true for all boundary colourings from C, we can conclude that C is Kempe closed and thus plausible.

It is clear that for a given permutation ϕ of length k and CBA C of degree k, we can easily obtain s-equivalent CBA C' that represents the cubic pole with an order of dangling edges permutated by ϕ (if C has a colour type M, C' will have a colour type $\phi(M)$). We can simply try all permutations of length k to see the whole s-equivalence class of any CBA. Our representative for each s-equivalence class will simply be the lexicographically smallest of those CBAs. We will denote the representative of CBA C under s-equivalence as $s(C)$. By trying all permutations of length k we can obtain $s(C)$ for any C.

We can therefore generate a list of s-equivalence representatives from the list of plausible CBAs by just calculating $s(C)$ for all C and storing each representative the first time we see it.

Similarly, as above, we will denote $k(C)$ and $sk(C)$ as representative of C under k-equivalence and sk-equivalence respectively. We will also denote $redComp(C)$ as the reduced complement of C.

Now, let C be some CBA. From the way Lemma 5 was proved, we can see that any plausible CBA disjoint from C is actually a subset of the reduced complement we are trying to find. Therefore, one of the ways we can find the reduced complement is to start with an empty set E and check for every plausible CBA if it is disjoint from C. If it is, we can add all its elements to E. The final iteration of E is the reduced complement we are trying to find. We can check the whole list of plausible CBAs and for each CBA C calculate its reduced complement D and if it is the first time we are seeing it, we can set $k(C) = C$ and put C into the list of representatives of k-equivalence classes.

To calculate sk-equivalence representatives, instead of $redComp(C)$, we have to calculate $s(redComp(C))$. This is because if two CBAs X and Y are sk-equivalent, we can permutate one of them by ϕ similarly as above to make them k-equivalent, in which case their reduced complement is the same. That means, without using ϕ, their reduced complements are s-equivalent, so $s(redComp(X))$ and $s(redComp(Y))$ are the same.

This is one of the algorithms we use, but as we can see, it does not work for colouring sets of degree 7 or higher, because we do not have a list of plausible colouring sets of degree 7. For colouring sets of higher degrees, we will use an algorithm that creates a complement of a given colouring set and afterwards checks if all boundary colourings satisfy Kempe closeness. If some of them do not, it throws them out and repeats the process, otherwise, it is finished (similarly as above, we need to also apply function $s()$ if we want to get a sk-equivalence representative instead). Therefore, we do not have a list of representatives for higher degrees, but we can still calculate $redComp(C)$, $s(C)$, $k(C)$ and $sk(C)$.

Besides ordinary unit tests, we also tested if those two algorithms produce the same result. We also tested Kempe closeness by taking some pairs of plausible sets and checking if their union is again plausible. Why this has to be true for correctly implemented Kempe closeness can be seen in the proof of Lemma 5.

For colouring sets of degree 6, the number of all plausible CBAs is 39962893. S-equivalence has 63049 classes and sk-equivalence has 170 classes (we did not explicitly calculate representatives of k-equivalence since we have no use for it).

Chapter 4

Realisations of sk-equivalence classes

We used several algorithms for realising sk-equivalence classes.

4.1 Expanding colouring sets

This algorithm was already done by my supervisor before I started working on this thesis. I only modified some parts of it, for example, to convert found CBAs to their sk-equivalence representative or to log data.

We will start with a queue containing a CBA representing an empty graph. Then we repeatedly modify a graph with one of these operations to obtain a new one:

1. Add a disconnected edge. (thereby creating a new $(k+2)$-pole from a k-pole)
2. Performing a junction of two dangling edges. (thereby creating a $(k-2)$-pole from a k-pole)
3. Add two new edges to a dangling edge. (thereby creating a $(k+1)$-pole from a k-pole)

While processing a CBA, we put all the different ways it can be modified at the end of the queue (clearly it suffices to do operations on CBAs, we do not need the whole graph). We can keep track of which CBAs we already found and only put the new ones in the queue. It can be seen, that this system of operations can produce any cubic pole (and can be easily proven by induction), so it will eventually generate every realisable CBA of degree 6. The problem is, it generates CBAs of arbitrarily large degrees and won't terminate. If we limit the degrees of CBAs we work with to 6 or lower, the algorithm finishes after a finite amount of time. This algorithm finds 40 out of 170 sk-equivalence classes. For $k \leq 5$ it already realises all sk-equivalence classes.

Interestingly, fully realising these classes tells us, that we do not need to run this program again, in case we find new CBA representatives of sk-equivalence classes for

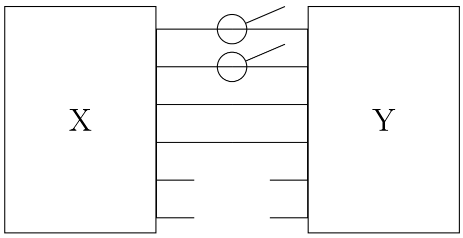

Figure 4.1: Combination of two colouring sets

$k = 6$. For this algorithm to realise a new class, we would either need a colouring set of degree 4 and add an edge to it or have a colouring set of degree 5 and add two edges to one of its dangling edges. Clearly, this algorithm covered all these cases when he found realisations for colouring sets of degrees 4 and 5.

4.2 Combining two colouring sets

We will be doing a more generalised junction between two already realised CBAs to find a new one.

For each dangling edge, we can do one of the following:

1. Do the junction with the dangling edge of the opposing cubic pole while adding an edge and a vertex.
2. Do the junction with the dangling edge of the opposing cubic pole.
3. Do nothing and let it be a dangling edge of a combined cubic pole.

A *Combination* is therefore similar to a junction, except instead of doing a junction between a pair of edges, we may also add a vertex and an edge. In figure 4.1 we can see a combination of colouring sets X and Y.

Clearly, we can combine two CBAs to obtain a new CBA without knowing which graphs they correspond to. Therefore, instead of combining graphs we will be combining CBAs in such a way that the resulting CBA corresponds to the graph we would get if we combined the initial graphs.

The simplest way to do a combination is to colour the first and second graph's dangling edges with all possible boundary colourings of corresponding CBAs and check if the combination is valid (i. e. if the operation of the first type connects the edges of the different colour and if the operation of the second type connects the edges of the same colour). If it is, we can add the resulting boundary colouring to the colouring set. This way we will get a new CBA.

Lemma 7 implies that instead of combining X and Y to obtain Z, we can combine $k(X)$ and $k(Y)$ to obtain Z' such that Z and Z' are k-equivalent (and also sk-equivalent, as sk-equivalence is a superset of k-equivalence). There clearly is a permutation ϕ such that permutating the "edges" of $k(X)$ with ϕ gives us $sk(X)$. The thing we need to realise is that permutating the "edges" of CBAs $k(X)$ and $k(Y)$ we want to combine the same way does not change the outcome of the combination. That means, permutating both $k(X)$ and $k(Y)$ with ϕ does not change the outcome, but changes the situation to a combination of $sk(X)$ and $k(Y)$. Instead of using $k(Y)$, we tried permutating $sk(Y)$ in all ways possible. The final version, and the way it is implemented in our code, is therefore taking two representatives from sk-equivalence, permutating edges of one of them and checking all possible ways they can be combined. Then we just transform the new CBA X we found into $sk(X)$.

This way, we are only using sk-equivalence representatives to find new ones.

The question is, which CBAs are best suited for combination? We can see that the CBA of degree 2 can always be reduced to just a single edge (Parity Lemma), so combining it with CBA X gives us just X. Therefore we will be connecting only CBAs of degrees 4, 5, 6, 7 (using CBAs of degrees 8 or higher is a lot harder as they have a greater number of colour types).

The way we are choosing which CBAs to combine is this: At the start of each iteration of our algorithm, we have an array of CBAs of degree 6 or 7 (this array can only contain sk-representatives). In the beginning, this array has all representatives of degree 6 we already found and 0 CBAs of degree 7. During the i-th iteration, we will try to combine the i-th CBA from the array with all previous CBAs from the array and with every sk-representative of degree 4 or 5. All found CBAs are pushed to the back of the array. If we do this for all CBAs, we are done.

The number of CBAs of degree 7 is much greater than the number of CBAs of lower degrees and we cannot check through it all, so we had to limit it to only CBAs with at most 20 colour types (out of 91, which CBAs of degree 7 can have). The motivation behind this is that combining two CBAs with a high number of "ones" (colour types) usually gives us a CBA with a high number of ones. From observations, CBAs with a high number of ones tend to be in bigger sk-equivalence classes, which means they are much easier to find. The chance to find something new is therefore bigger if we use CBAs with a small number of ones. Additionally, the computing time of calculating a combination is roughly proportional to each CBA's number of ones, so we can compute a combination for these CBAs faster.

There is also another way to filter CBAs of degree 7. Some cubic 7-poles are a union of a cubic 2-pole and a cubic 5-pole or a cubic 3-pole and a cubic 4-pole. These cannot help with realising new colouring sets (we can check that combining such cubic 7-poles with something is like doing two different combinations in sequence, one for each of its components). There is a small amount of CBAs that represent such cubic 7-poles, so we can check all of them, calculate sk-equivalent representative for them and not use those representatives in our algorithm.

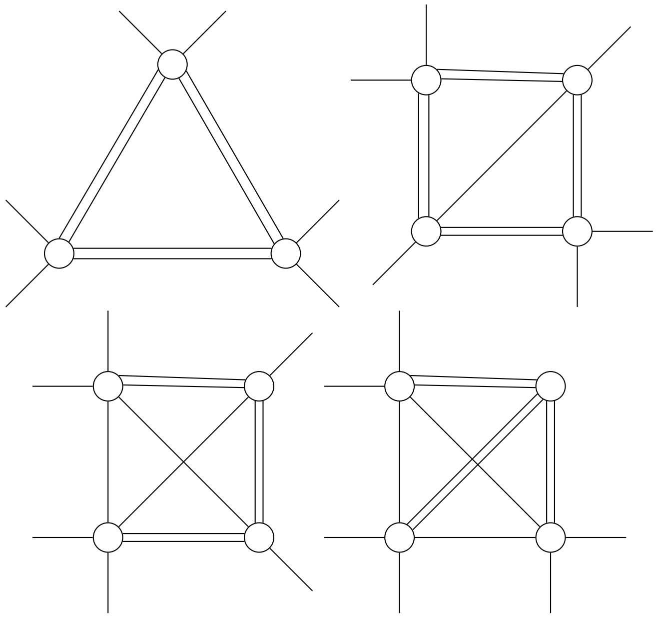

Figure 4.2: Various ways to connect 3 or 4 CBAs together

4.3 Combining several colouring sets

Another way to combine CBAs is to combine more than 2 of them in such a way, that the resulting graph cannot be made by a sequence of combinations from the last algorithm. In Figure 4.2 we can see all the ways 3 or 4 CBAs can be combined together, without loss of generality (we are using a regular junction). Each vertex of this "supergraph" represents a cubic pole and each edge incident to the vertex represents dangling edges of that cubic pole.

We can be verified by a simple brute force, that these are all possible supergraphs of orders 3 and 4, other supergraphs would have neighbouring vertices whose connection would already be covered by an earlier algorithm or could be split into two parts such that the edges between them are a junction we already tried (in which case it would make sense to just try doing one part of the supergraph).

For the supergraph of degree 3, we tried various symmetric cases (for example 3 copies of the same CBA). For other supergraphs, the number of possible combinations is too large, so we also created an algorithm that just takes random CBAs, permutates the edges randomly and calculates the resulting CBA. We also tried supergraphs with $k \geq 5$ for a while, but the computing time increases exponentially with the number of vertices, so we stuck to doing as much junctions as we can described by Figure 4.2.

4.4 Comparison with previous work

We compared our findings with work done by Karabáš, Máčajová and Nedela [5]. In their work, they described a partially ordered set of sk-equivalence classes. They define that $A \leq B$ if and only if we can find s-equivalent A' to A, such that $A' \subseteq B$. We verified that there are 38 minimal CBAs (as stated in their work), ignoring empty CBA. In Table 4.1, we can see that we managed to realise the first 9 of them (they achieved the same result).

Their notation	**Our notation**	**Sk-equivalence representative**	**Found**
1112539137	42991619	1107394561	Yes
219	126	55306	Yes
995328	1006657536	1711325184	Yes
1782	3576	55412	Yes
983094	1006633368	1711282186	Yes
983277	1006633446	1711282292	Yes
1186463953	123207795	1682147339	Yes
259394321	465311250	2024244548	Yes
1186467169	123214627	1682147445	Yes
15275	30542	54770	No
986027	1006638926	1711281650	No
1003880	1006674724	1711320160	No
1003955	1006674778	1711323848	No
1005470	1006677724	1711323830	No
107940770	1033377608	1762625594	No
108429655	1536688826	2097762112	No
108429708	1536688836	2097763344	No
108431226	1536692028	62729532	No
108441953	1536713506	2097758918	No
108442042	1536713564	62731884	No
108454116	1536737760	2097777690	No
108456866	1536743240	62780744	No
259391628	465305796	2024221184	No
259394506	465311340	465311340	No
259407114	465336844	465336844	No
259415138	465353000	465353000	No
1112539354	42991741	1107384331	No
1112542108	42997461	1107384437	No
1112554410	43022157	1107383795	No
1113522412	1049625061	2084702921	No
1113525162	1049630541	2084702903	No
1113543090	1049666393	2084751763	No
1113545460	1049671153	49724729	No
1113545730	1049671689	2084750357	No
1186479009	123238211	1682146803	No
1186490916	123262337	1682122545	No
112984273	257425522	1758073664	No
127372309	2054690962	2102003504	No

Table 4.1: 38 minimal CBAs

Conclusion

In this thesis, we were trying to find out which colouring sets are realisable with the purpose of better understanding k-decomposition of snarks. We showed the connection between cubic poles, colouring sets and k-decomposition. We used Kempe closeness and Parity lemma to define the necessary condition for realisability. We also used c-equivalence and s-equivalence and introduced a new k-equivalence. The unifying sk-equivalence helped us divide colouring sets into classes of colouring sets which are for our purposes equivalent. For $k \leq 6$ we managed to create a list of representatives of s-equivalence and sk-equivalence. The most notable was the sk-equivalence for colouring sets of degree 6, for which we enumerated all 170 representatives.

We aimed to realise these representatives by doing various approaches. Our first algorithm iteratively expands an empty colouring set by adding an extra edge to find new colouring sets. Our other algorithms combine already realised cubic poles in various ways. Using sk-equivalence we managed to search for new colouring sets more effectively. Overall we realised 115 out of 170 sk-equivalence classes. Classes we found represent roughly 99.9% of all plausible colouring sets.

We also compared and verified some findings about colouring sets from previous work done by Karabáš, Máčajová, and Nedela. By identifying the 170 sk-equivalence classes, we reduced the problem of k-decomposition, to only solving it for one colouring set from each class. If further research proves the remaining 55 classes are unrealisable, the problem is further reduced to 115 classes. We also constructed a partially ordered set out of all 170 classes, which may help with identifying irreducible colouring sets, which are strictly better for performing a k-decomposition than reducible colouring sets.

All our algorithms for realising colouring sets could be modified to cover a wider range of scenarios. The first algorithm could check cubic k-poles for higher k, which would require significantly more computing power. Similarly, in the second algorithm, we could restrict the number of ones cubic 7-poles can have to a higher number. Supergraphs of orders 3 and 4 were also not fully examined and could be checked for all combinations of cubic poles. Future studies might also explore specific supergraphs of even higher orders that, for some reason, could offer better chances of success.

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