Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

The Cycle Double Cover Conjecture and Related Conjectures

Bachelor Thesis

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	doc. RNDr. Edita Mačajová PhD.

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Annotation: In 1973 Szekeres formulated the cycle double cover conjecture that states that every bridgeless graph has a collection of cycles containing every edge exactly twice. This conjecture is equivalent to the assertion for cubic bridgeless non-colorable graphs. Despite the enormous effort of researchers in this field, the conjecture remains open. To better understand the problem, researchers have begun to study related conjectures. In this thesis, we will pursue our own conjecture, a strengthened version of the original conjecture for cubic graphs with the cyclic connectivity at least 4 and the girth at least 5. Our conjecture states that for such graphs exists a cycle double cover consisting of at most 6 cycles consisting of induced circuits as well as a cycle double cover consisting of at most 7 induced cycles. We will try to prove (or disprove) our conjecture for some infinite graph classes and discover new facts about cycle double covers.

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Názov: The Cycle Double Cover Conjecture and Related Conjectures
Hypotéza o dvojitom pokrytí cyklami a príbuzné hypotézy

Anotácia: V roku 1973 Szekeres sformuloval hypotézu o dvojitom pokrytí cyklami, ktorá hovorí, že každý bezmostový graf má kolekciu cyklov obsahujúcich každú hranu presne dvakrát. Táto hypotéza je ekvivalentná redukcii pre bezmostové nezafarbiteľné kubické grafy. Napriek enormnej snahe výskumníkov v tejto oblasti zostáva táto hypotéza otvorená. Pre lepšie pochopenie problému sa začali výskumníci zaoberať príbuznými hypotézami. V tejto bakalárskej práci sa budeme zaoberať našou vlastnou hypotézou, ktorá je zosilnením pôvodnej hypotézy a je vyslovená pre grafy, ktoré majú cyklickú súvislosť aspoň 4 a obvod aspoň 5. Naša hypotéza hovorí, že pre takéto grafy existuje dvojité pokrytie cyklami, ktoré sa skladá z najviac 6 cyklov zložených z indukovaných kružníc, a tiež dvojité pokrytie cyklami tvorené najviac 7 indukovanými cyklami. Budeme sa snažiť dokázať (alebo vyvrátiť) našu hypotézu pre niektoré nekonečné triedy grafov a objaviť nové poznatky o dvojitých pokrytiach cyklami.

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Abstrakt

V tejto bakalárskej práci sa zaoberáme našou vlastnou hypotézou o dvojitom pokrytí grafu cyklami. Na začiatku uvádzame dôležité definície, ktoré používame v celej práci spolu s pozorovaniami a známymi poznatkami o dvojitých pokrytiach grafu cyklami. Potom uvádzame nové poznatky, ktoré sme objavili o hranových 2-rezoch, netriviálnych hranových 3-rezoch a trojuholníkoch vo vzťahu k dvojitým pokrytiam grafu cyklami. Tiež vyslovíme našu hypotézu a poskytneme dôkazy na niektorých nekonečných triedach grafov, ako napríklad Issacsové snarky. Na záver popisujeme implementačné detaily nášho softvéru, zahrňajúc, ako sme reprezentovali grafy, algoritmus, ktorý sme používali na hľadanie dvojitých pokrytí grafu cyklami spolu s niektorými metódami, pomocou ktorých sme našli dvojité pokrytia cyklami, ktoré sme použili v dôkazoch našej hypotézy.

Kľúčové slová: Hypotéza o dvojitom pokrytí cyklami, CDC, indukovaný cyklus, kubické grafy, snark

Abstract

In this thesis, we pursue our own conjecture regarding cycle double covers. Firstly, we provide important definitions we use throughout the thesis, as well as observations and known facts about cycle double covers. Later, we state the new facts we discovered about 2-edge cuts, nontrivial 3-edge cuts, and triangles with regard to cycle double covers. We also state our conjecture and provide proofs of our conjecture for some infinite graph families such as Issacs snarks. Lastly, we describe implementation details of our software, including how we represented graphs, the algorithm we used for finding cycle double covers in addition to some methods that helped us to find the cycle double covers that we used in proofs of our conjecture.

Keywords: The Cycle Double Cover Conjecture, CDC, induced cycle, cubic graphs, snark

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Introduction

In 1973, Hungarian-Australian mathematician George Szekeres [1] formulated the cycle double cover conjecture that states that every bridgeless graph has a collection of cycles containing every edge exactly twice. Our thesis focuses on our own conjecture, which is a strengthened version of the original cycle double cover conjecture. The cycle double cover conjecture is equivalent to the assertion for bridgeless cubic graphs [4]. Therefore, we tried to observe connections between cycles passing through end vertices of edges of cubic graphs in cycle double covers.

Our conjecture states that for every bridgeless cubic graph with the length of its shortest circuit at least 5 and without 2-edge cuts and nontrivial 3-edge cuts, there is a circuit double cover comprised of chordless circuits. In cubic graphs, we can merge circuits into cycles without a vertex in both circuits. We suggest that in a circuit double cover consisting of chordless circuits, we can combine these circuits into at most six cycles. There might be an edge between the chordless circuits that comprise a cycle, so we conjecture that for the considered graphs also exists a cycle double cover with at most seven induced cycles.

The cycle double cover conjecture is already proven for some graph families, such as bridgeless cubic graphs with chromatic index 3 [4]. By Vizing's theorem [9], cubic graphs have either chromatic index 3 or 4. Hence, the cycle double cover conjecture remains open for the latter graphs. Our conjecture includes such graphs with the exception of those with short circuits and small edge cuts. Our thesis aims to prove our conjecture on some infinite graph families. We also intend to discover new facts with regard to the cycle double covers that fulfill our conjecture with the help of the software that we created to find cycle double covers of bridgeless cubic graphs. Firstly, we provide important definitions, state the cycle double cover conjecture, and present facts in regard to the conjecture. We provide the results we discovered in addition to proofs of our conjecture on some infinite graph families. Lastly, we present implementation details of our software and describe some methods that we used to help us prove our conjecture.

Chapter 1

Preliminaries

In the first part of this chapter, we state necessary definitions that we use throughout the thesis. The second part provides a formulation of the cycle double cover conjecture, relevant definitions, and known results for a deeper understanding of the problem.

1.1 Definitions

This section presents formal definitions in graph theory. More can be found in *Graph theory* [2].

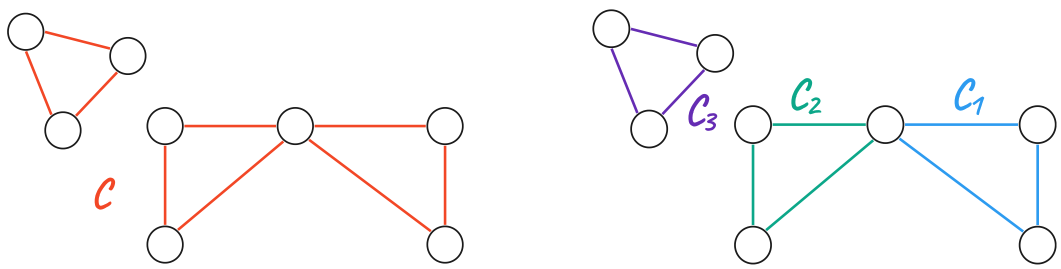
Definition 1.1.1 *The degree of a vertex is the number of neighbors of the vertex.*

Definition 1.1.2 *A k-regular graph is a graph in which all vertices have the same degree k. A 3-regular graph is called a cubic graph.*

Definition 1.1.3 *A circuit is a connected 2-regular graph.*

Definition 1.1.4 *A cycle is a graph in which all vertices have an even degree.*

Equivalently we can define a cycle as a collection of edge-disjoint circuits, as shown in the figure below. Note that a circuit is also a cycle.



(a) An example of a cycle

(b) Decomposition of the cycle

Figure 1.1: The cycle C is a collection of three circuits, C_1, C_2, and C_3.

Definition 1.1.5 *The length of a circuit is the number of its edges.*

Definition 1.1.6 *The girth of a graph is the minimum length of a circuit in the graph.*

Definition 1.1.7 *A graph $G = (V, E)$ is a subgraph of $G' = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$, written as $G \subseteq G'$. If $G \subseteq G'$, then G' is a supergraph of G.*

Definition 1.1.8 *A chord of a circuit is an edge of a supergraph of the circuit that is not part of the circuit but is incident with two vertices of the circuit. A chordless circuit is also called an induced circuit.*

Definition 1.1.9 *Let G be a graph. A cycle in G is induced if it consists of induced circuits, and G does not contain any edge such that both end vertices are contained in the cycle, but the edge is not part of the cycle.*

Definition 1.1.10 *A k-edge cut of a connected graph G is a set of k edges $S \subseteq E(G)$ such that the graph $G - S$ is a disconnected graph. A 1-edge cut is called a bridge.*

Definition 1.1.11 *A nontrivial k-edge cut is a k-edge cut of a connected graph such that all edges of the k-edge cut are not incident to the same vertex.*

Definition 1.1.12 *Let G be a connected graph. An edge cut S of G is cycle-separating if both components of the graph $G - S$ contain a cycle.*

Definition 1.1.13 *A graph is cyclically k-edge-connected if every cycle-separating edge cut of the graph has at least k edges.*

Definition 1.1.14 *An edge coloring of a graph G is a map c: $E(G) \to C$ with $c(e_1) \neq c(e_2)$ for any adjacent edges e_1, e_2. The elements of set C are called available colors.*

Definition 1.1.15 *The chromatic index of a graph G is the smallest integer k such that there exists an edge coloring of G c: $E(G) \to \{1, 2, ..., k\}$. The chromatic index of G is denoted by $\chi'(G)$.*

Definition 1.1.16 *A snark is a bridgeless cubic graph with chromatic index 4. A nontrivial snark is a snark that is cyclically 4-edge-connected and has girth at least 5.*

Definition 1.1.17 *The subdivision of an edge uv by a vertex w is the removal of the edge uv from the graph and the addition of the vertex w along with the new edges uw and wv.*

Figure 1.2: The subdivision of the edge uv by the vertex w

Definition 1.1.18 *The smoothing of a vertex w of degree 2 in regards to the edges uw, wv is the removal of the vertex w along with the edges uw, wv and the addition of the edge uv.*

Figure 1.3: The smoothing of the vertex w in regard to the edges uw, wv

Definition 1.1.19 *Assume that a vertex u with degree $k \geq 3$ has adjacent vertices $v_1, ..., v_k$. An inflation of the vertex u is the removal of the vertex u along with the edges $uv_1, ..., uv_k$ and the addition of vertices $u_1, ..., u_k$ along with the edges $u_1u_2, u_2u_3, ..., u_ku_1$ and $u_1v_1, ..., u_kv_k$.*

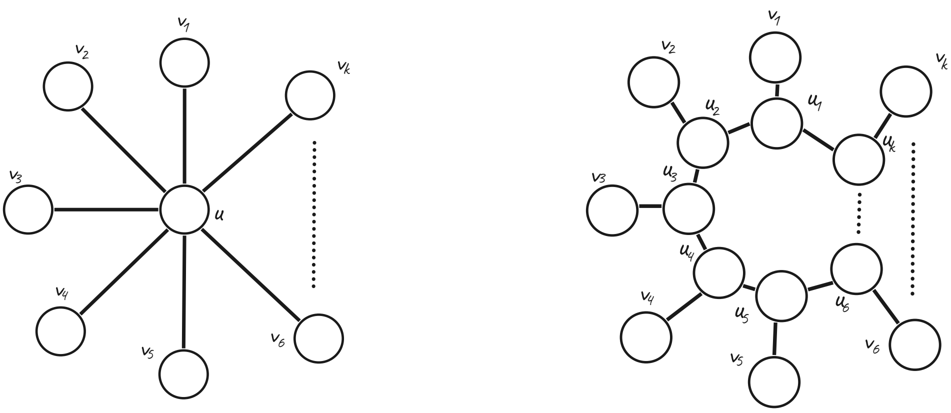


Figure 1.4: An inflation of a vertex u

Definition 1.1.20 *Let G_1 and G_2 be two bridgeless cubic graphs, with $u \in V(G_1)$ and $v \in V(G_2)$ such that the adjacent vertices of u are u_1,u_2,u_3, and those adjacent to v are v_1,v_2,v_3. A 3–cut-connection on u and v is a graph operation consisting of constructing the new graph $(G_1 - u) \cup (G_2 - v) \cup \{v_1u_1, v_2u_2, v_3u_3\}$.*

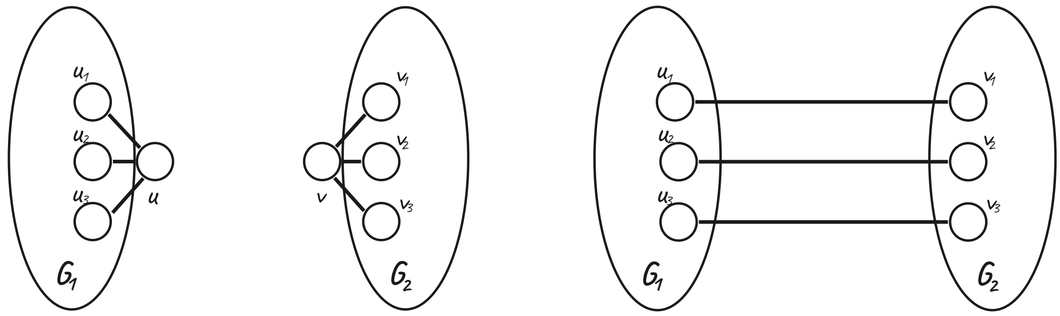


Figure 1.5: A 3-cut-connection on u and v

1.2 The Cycle Double Cover Conjecture

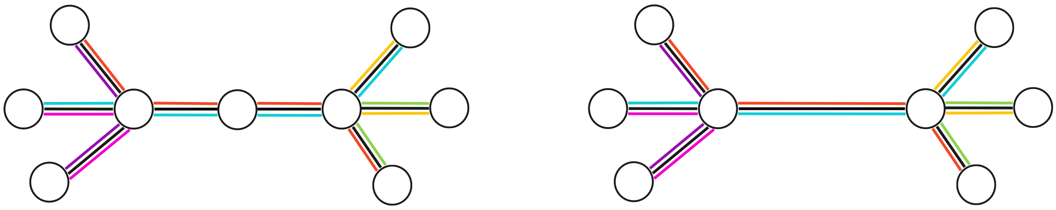
In this section, we provide a formulation of the cycle double cover conjecture and state important definitions in regard to the conjecture as well as known facts.

Definition 1.2.1 *A cycle (circuit) double cover of a graph is a collection of cycles (circuits) that contains every edge of the graph exactly twice. We abbreviate the cycle double cover to CDC. A k-CDC is a cycle double cover with at most k cycles.*

As we stated all necessary definitions, we present the formulation of the cycle double cover conjecture.

Conjecture 1.2.2 *(The Cycle Double Cover Conjecture, Szekeres [1]) Every bridgeless graph has a cycle double cover.*

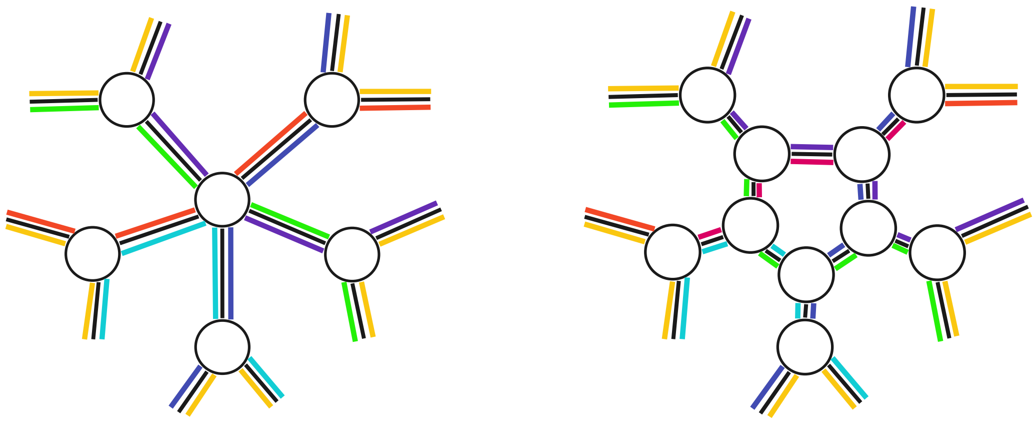
The conjecture can be easily reduced to bridgeless cubic graphs [4] as follows: Assume that we have a bridgeless graph G. We want to transform G into a bridgeless cubic graph and show that G has a CDC if the obtained bridgeless cubic graph has a CDC. The graph G cannot have any vertex with degree 1 as it implies that G has a bridge. Let us consider the vertices of degree 2. We can eliminate these vertices by the smoothing of these vertices, as shown in the figure below.



(a) Before the smoothing (b) After the smoothing

Figure 1.6: The smoothing of a vertex with degree 2 and the impact on a CDC

Observe that if the obtained graph has a CDC, we can easily reverse the smoothing of a vertex by the subdivision of the affected edge by the considered vertex and get a CDC of the original graph. If the graph contains only the vertices with degree 2, then the graph is a cycle itself, and it can be covered twice by itself. As we want a cubic graph, we can ignore the vertices with degree 3. Lastly, we need to process the vertices of degree at least 4. Assume we have a vertex with degree $k \geq 4$. We transform the vertex into a k-gon by any inflation of the vertex, and, as a result, we get new vertices with degree 3. Let us assume that we have a CDC of the obtained k-gon.



(a) Before an inflation (b) After the inflation

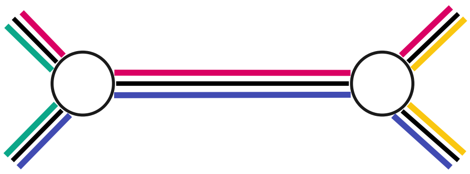
Figure 1.7: An example of the impact of an inflation of a vertex with degree 5 on a CDC

As one easily notices, we can reverse the inflation of the vertex and get a CDC of the original graph based on the CDC of the obtained graph. As a result, we can get a cubic graph from any graph that is not a cycle itself, such that the original graph has a CDC if the obtained cubic graph has a CDC. Therefore, from now on, we will only consider cubic graphs.

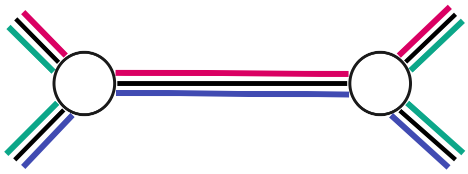
Proposition 1.2.3 *In cubic graphs, three cycles pass through each vertex in a CDC.*

We will distinguish between the edges whose end vertices have the same sets of cycles passing through them and the edges that have different sets of cycles passing through their end vertices. Note that since edges are covered by two cycles, these two cycles contain both end vertices, so the third cycle, which does not cover an edge but contains its end vertices, determines whether the cycles containing the end vertices of the edge are the same. We will refer to this decisive cycle as *the third cycle* of the edge.

Definition 1.2.4 *Let G be a bridgeless cubic graph with a CDC $\mathcal{C}$. If the sets of cycles passing through end vertices of an edge are different in $\mathcal{C}$, we call such an edge a strong edge in $\mathcal{C}$. Otherwise, the edge is weak in $\mathcal{C}$.*



(a) A strong edge



(b) A weak edge

Figure 1.8: The difference between a strong edge and a weak edge.

Definition 1.2.5 *A CDC $\mathcal{C}$ in a graph G is semi-induced if each cycle in $\mathcal{C}$ is comprised of induced circuits.*

Assume that a bridgeless cubic graph has a semi-induced CDC. If we decomposed all cycles of the CDC into circuits, then all edges would be strong in the obtained circuit double cover.

Definition 1.2.6 *A CDC $\mathcal{C}$ in a graph G is induced if each cycle in $\mathcal{C}$ is induced.*

Note that if a cycle is induced, it implies that it consists of induced circuits such that there is no edge between the circuits of the cycle. As a result, if a CDC is induced, then it is also semi-induced, and all edges are strong in the CDC as well as in the circuit double cover obtained by decomposition of the CDC into circuits. One easily observes that a circuit double cover consisting of induced circuits is also an induced CDC as there are no weak edges in such a circuit double cover.

We now provide a crucial connection between chords and weak edges in circuit double covers.

Lemma 1.2.7 *Let G be a bridgeless cubic graph with a circuit double cover $\mathcal{C}$. An edge $uv \in E(G)$ is weak in $\mathcal{C}$ if and only if uv is a chord of a circuit in $\mathcal{C}$.*

Proof: Assume that uv is weak in $\mathcal{C}$. Since the edge uv is covered by two circuits, C_1 and C_2 in $\mathcal{C}$, C_1 and C_2 contain the vertex u as well as the vertex v. As circuits passing through the vertices u and v must be the same, there must be a circuit C_3 in $\mathcal{C}$ that contains both u and v. But C_3 does not cover the edge uv, so uv is a chord of the circuit C_3.

Assume that uv is a chord of a circuit C in $\mathcal{C}$. As a consequence, C contains both vertices u, v but does not cover the edge uv. The edge uv must be covered by two circuits in $\mathcal{C}$. These two circuits contain both vertices u, v. Therefore, circuits passing through the vertex u are the same as circuits passing through the vertex v. Hence, the edge uv is weak. □

Corollary 1.2.8 *The number of strong edges in a circuit double cover of a cubic graph is the number of edges that are not chords of any circuits in the circuit double cover.*

One easily observes that a weak edge in a cycle double cover is either a chord of a circuit obtained by decomposition of a cycle, or an edge between circuits of the same cycle.

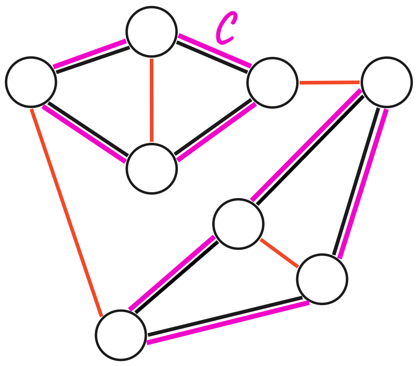
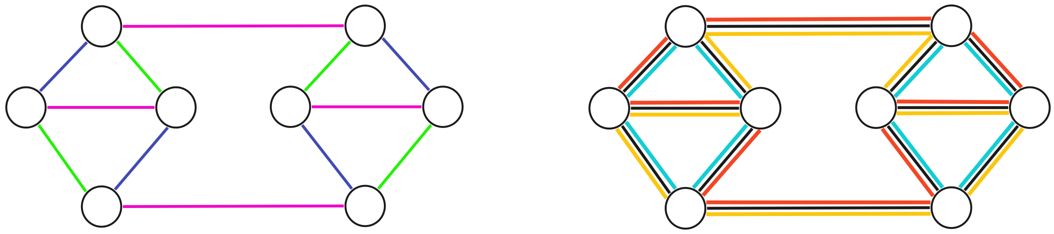


Figure 1.9: Potential weak edges are denoted by the red color

The following theorem belongs to the folklore with regard to cycle double covers.

Theorem 1.2.9 *Let G be a bridgeless cubic graph with chromatic index 3. Then G has a 3-CDC.*

Proof: The cubic graph G has chromatic index 3, so the edges of G can be colored with three colors such that adjacent edges have different colors. Since G is cubic, each vertex of G is incident with three edges that have three different colors. If we take a combination of two colors, the edges form a cycle since they form a 2-regular graph. We get three cycles as there are three 2-combinations of these three colors. Every color (thus every edge) is a part of two 2-combinations. Therefore, the cycles cover each edge precisely two times.



(a) A cubic graph with chromatic index 3 (b) The cubic graph with a 3-CDC

□

Observe that each edge of a graph is weak in a 3-CDC as these three cycles contain each vertex of the graph.

Chapter 2

Results

In this chapter, we present the results we discovered regarding cycle double covers.

2.1 Results regarding 2-edge cuts

Theorem 2.1.1 *Let G be a bridgeless cubic graph with a 2-edge cut and a circuit double cover $\mathcal{C}$. Then the 2-edge cut edges are strong in $\mathcal{C}$.*

Proof: Let $S = \{e_1, e_2\}$ be a set of the edges of the 2-edge cut such that $G - S$ has two components, G_1 and G_2.

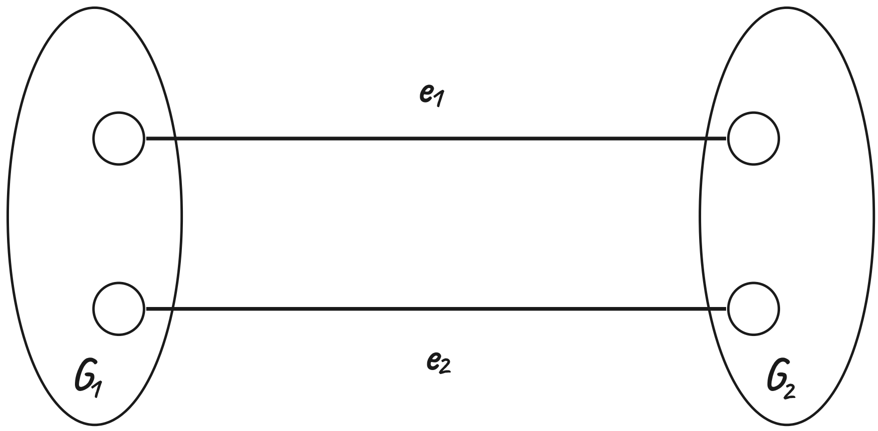


Figure 2.1: A graph G with the 2-edge cut $\{e_1, e_2\}$

Let G have a circuit double cover $\mathcal{C}$. Then the edge e_1 is contained in precisely two circuits, C_1 and C_2. Without loss of generality, let the circuit C_1 start in the subgraph G_1. The circuit C_1 passes through the edge e_1. Since $S = \{e_1, e_2\}$ is a 2-edge cut separating the subgraphs G_1 and G_2, every circuit passing from the subgraph G_1 to the subgraph G_2 must return from the subgraph G_2 back to the subgraph G_1. The only way back is through the edge e_2. Hence, the circuit C_1 passes through the edge e_2 as well.

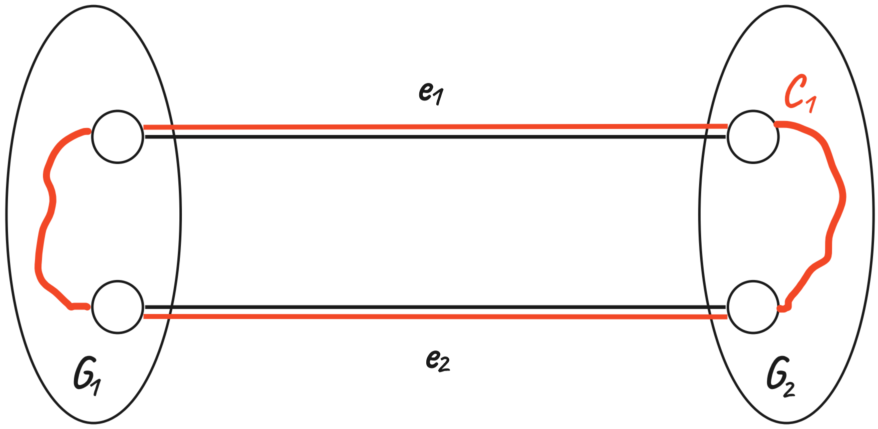


Figure 2.2: The graph G with the circuit C_1

Analogously the circuit C_2, starting in the subgraph G_2 and passing through the edge e_1, must also pass through the edge e_2.

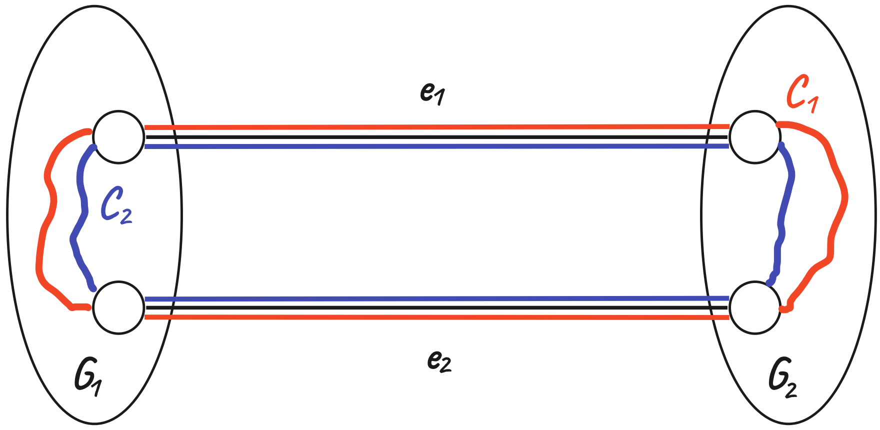


Figure 2.3: The graph G with the circuits C_1 and C_2

The edges e_1 and e_2 are now covered by these two circuits. Therefore, no other circuit in $\mathcal{C}$ passes through them. As the subgraphs G_1 and G_2 are disjoint, the third circuits, C_3 and C_4, passing through the end vertices of the edge e_1, are different from one another. Hence, the edge e_1 is strong in $\mathcal{C}$. Similarly, different circuits, C_5 and C_6, pass through the end vertices of the edge e_2. Therefore, the edge e_2 is strong in $\mathcal{C}$ as well. Note that the circuits passing through the end vertices of the edges e_1 and e_2 in the same subgraph might be the same. $\square$

Theorem 2.1.2 *Let G be a bridgeless cubic graph with a 2-edge cut set $S = \{e_1, e_2\}$. Assume that G' is obtained from G by the subdivision of the edges e_1 and e_2 by vertices k_1 and k_2, respectively, and the addition of the edge k_1k_2. Then the edge k_1k_2 is weak in any cycle double cover.*

Proof: Assume that the 2-edge cut set $S = \{e_1, e_2\}$ of the graph G connects two components of the graph $G - S$, G_1 and G_2, and the edges e_1 and e_2 split by their subdivisions into edges e_1', e_1'' and e_2', e_2'', respectively.

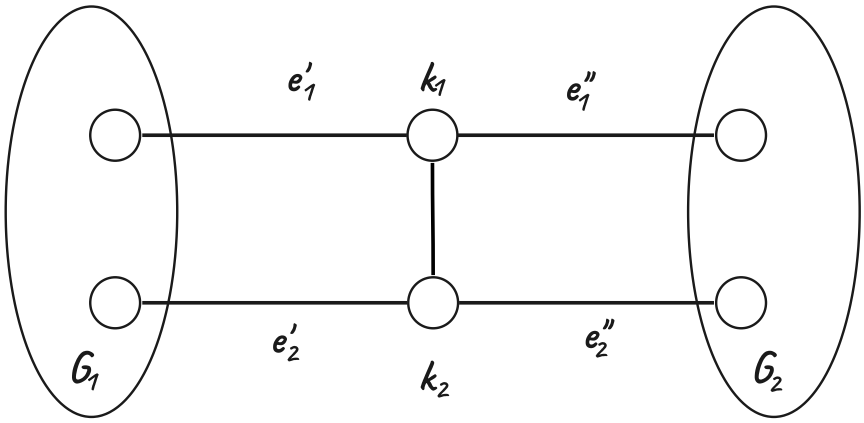


Figure 2.4: The obtained graph G'

Assume that the graph G' has a cycle double cover $\mathcal{C}$. In $\mathcal{C}$, each edge is covered by exactly two cycles. As one readily observes, any cycle covering the edge k_1k_2 cannot pass through both edges e_1', e_1'' or e_2', e_2''. Therefore, a circuit C that is part of a cycle in $\mathcal{C}$ passes through the edges e_1', e_1'' and, subsequently, through the edges e_2'', e_2', but not through the edge k_1k_2. As a result, the edge k_1k_2 is a chord of the circuit C. Hence, the edge k_1k_2 is weak.

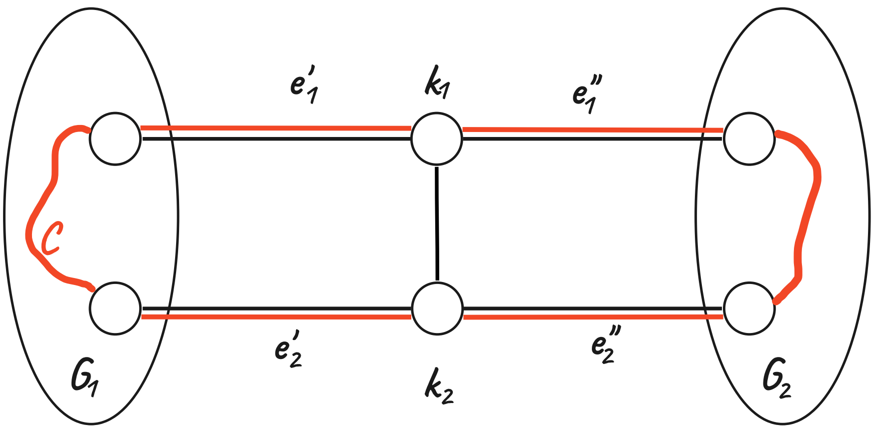


Figure 2.5: The circuit C in the graph G'

□

2.2 Results regarding nontrivial 3-edge cuts

Theorem 2.2.1 *Assume that G_1 and G_2 are bridgeless cubic graphs, $k_1 \in V(G_1)$ and $k_2 \in V(G_2)$. Let G be a graph obtained by a 3-cut-connection on k_1 and k_2.*

(a) If G_1 and G_2 have a semi-induced 6-CDC, then so has G.

(b) If G_1 and G_2 have an induced 7-CDC, then so has G.

It is already proven that if G_1 and G_2 have a cycle double cover, then G has a cycle double cover as well. Nevertheless, we provide a proof.

Proof: Assume that the graphs G_1 and G_2 have a cycle double cover. Firstly, we prove that the obtained graph G also has a cycle double cover. Assume that cycles C_1, C_2, C_3 pass through the vertex k_1 in the graph G_1. Similarly, assume that cycles C_4, C_5, C_6 pass through the vertex k_2 in the graph G_2. Assume that the neighbors of the vertex k_1 are vertices v_1, v_2, v_3, the neighbors of the vertex k_2 are vertices v_4, v_5, v_6, and the 3-cut-connection was performed in such a way that the edges v_1v_4, v_2v_5, v_3v_6 were added.

Figure 2.6: The graphs G_1 and G_2 with the marked vertices k_1 and k_2

Firstly, remove the vertices k_1 and k_2 with their incident edges.

Figure 2.7: The graphs G_1 and G_2 with the removed vertices k_1 and k_2

Then, we obtain the graph G by the addition of the edges $e_1 = (v_1, v_4), e_2 = (v_2, v_5)$ and $e_3 = (v_3, v_6)$.

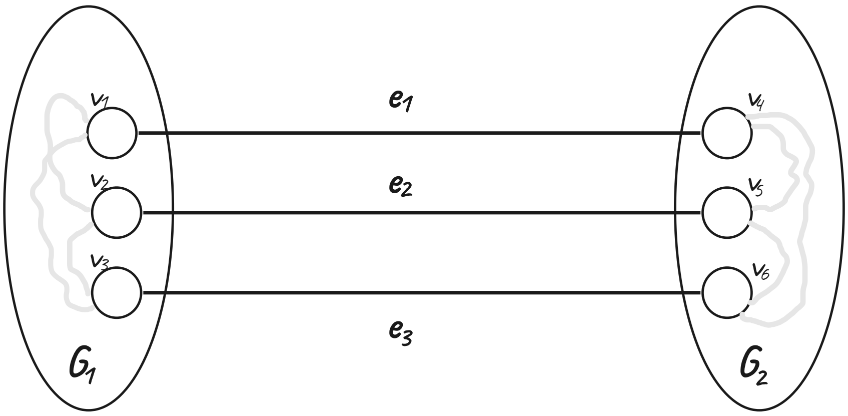


Figure 2.8: The graph G obtained from the graphs G_1, G_2

We need to cover each of the edges e_1, e_2, e_3 with two cycles in addition to the edges that are not covered with two cycles since the cycles $C_1, ..., C_6$ no longer exist. We can make use of the remaining parts of the cycle C_1, especially the part from the vertex v_1 to the vertex v_2. Likewise, we can use the remaining parts of the cycle C_5, especially the part from the vertex v_4 to the vertex v_5. Note that in the graph G we added the edges $e_1 = (v_1, v_4)$ and $e_2 = (v_2, v_5)$. Therefore, there exists a cycle C'_1 such that it covers the edges e_1, e_2 as well as the edges that were previously covered by the remaining parts of the cycles C_1 and C_5. Analogously, there exists a cycle C'_2 such that it covers the edges e_1, e_3 in addition to the edges that were previously covered by the remaining parts of the cycles C_2, C_4. Lastly, there also exists a cycle C'_3 covering the edges e_1, e_3 along with the edges previously covered by the remaining parts of the cycles C_3, C_6. Hence, all edges of the graph G are covered by two cycles because the cycle double cover of the parts that were not affected stays the same. We proved that the obtained graph G has a cycle double cover.

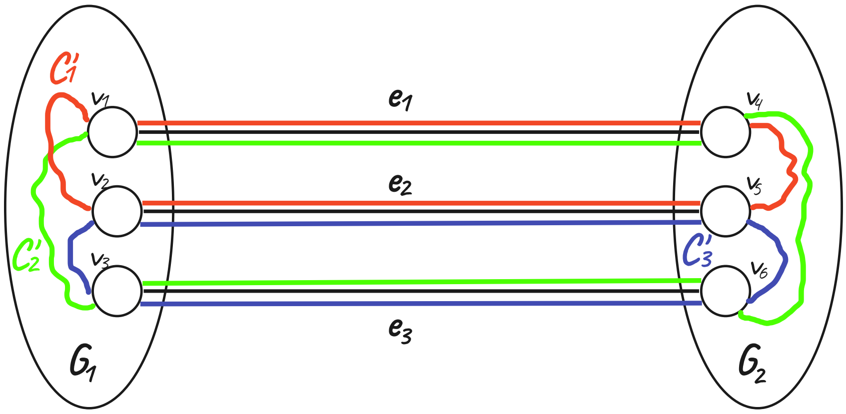


Figure 2.9: The graph G with the cycle double cover

(a) Assume that the graphs G_1 and G_2 have a semi-induced 6-CDC. As we observed, the obtained graph G has a CDC with the three cycles C_1', C_2', C_3' covering the edges of the newly formed nontrivial 3-edge cut. As a result, the cycle double cover consists of the three considered cycles C_1', C_2', C_3' and at most three unaffected cycles in both subgraphs G_1' and G_2', corresponding to the graphs G_1 and G_2. Let the unaffected cycles in G_1' and G_2' be C_4, C_5, C_6 and C_7, C_8, C_9, respectively. We want to prove that there is a cycle double cover of G that is a semi-induced 6-CDC.

Firstly, we want to show that the CDC is semi-induced. As one readily observes, the unaffected cycles in both subgraphs remain semi-induced. We want to prove that the cycles C_1', C_2', C_3' are also semi-induced. Each of them consists of two of the newly added edges e_1, e_2, e_3 and the unaffected parts that formed semi-induced cycles, meaning that there were no weak edges in the unaffected parts. If we take the edges e_1, e_2, e_3, we know that each of the cycles C_1', C_2', C_3' covers exactly two of the edges e_1, e_2, e_3. The edge e_3 that is not covered with C_1' would be weak if C_1' contained the vertex v_3 or v_6. Suppose that C_1' contains the vertex v_3. It means that the original C_1 contained v_3. Consequently, the edge k_1v_3 was weak. Thus, a contradiction to the semi-induced 6-CDC. The same argument holds for the cycles C_2' and C_3'.

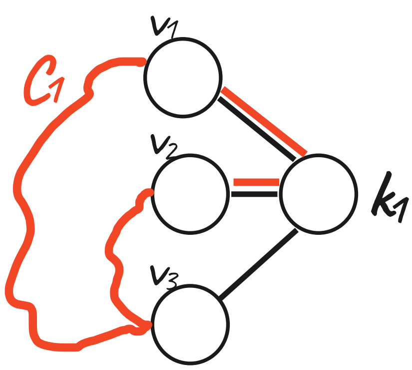


Figure 2.10: The cycle C_1 with the weak edge k_1v_3

We proved that the cycle double cover of G is semi-induced. Now we show that it can contain at most six semi-induced cycles. As both graphs G_1 and G_2 had the 6-CDCs, the obtained cycle double cover consists of at most nine semi-induced cycles $C_1', C_2', C_3', C_4, ..., C_9$. As the subgraphs G_1' and G_2' are disjoint, we can merge the cycles that occur in only one of the subgraphs G_1', G_2' into larger cycles since they are disjoint as well. Note that C_1', C_2', C_3' occur in both subgraphs G_1', G_2'.

In both subgraphs, G_1' and G_2', are at most three distinct semi-induced cycles C_4, C_5, C_6 and C_7, C_8, C_9, respectively, and we need to get at most three larger semi-induced cycles apart from C_1', C_2', C_3'. If both subgraphs contain exactly three other cycles, we can just randomly pair the cycles and merge them into three larger semi-induced cycles. Observe that if there are in total less than six cycles to join, we can keep some cycles unchanged and still get at most six cycles. Hence, we proved that the obtained graph G has a semi-induced 6-CDC.

(b) Assume that the graphs G_1 and G_2 have an induced 7-CDC. We observed before that the obtained graph G has a cycle double cover as well. We want to show that there is a cycle double cover that is an induced 7-CDC. Suppose the considered CDC contains at most four unaffected induced cycles $C_4, ..., C_7$ and $C_8, ..., C_{11}$ in both subgraphs, G_1' and G_2', respectively. It also contains the cycles C_1', C_2', C_3' that cover the 3-edge cut.

One easily observes that the unaffected cycles remain induced. The cycles C_1', C_2', C_3' were derived from induced cycles, and they cover exactly two edges of the cut edges e_1, e_2, e_3. As a result, a weak edge could only be the uncovered cut edge. We observed that if the uncovered cut edge were a weak edge, there would be a weak edge in the original CDC. Thus, a contradiction to the induced 7-CDC.

We proved that the obtained CDC is an induced CDC. Now we show that it can contain at most seven induced cycles. Since both subgraphs had the 7-CDCs, the obtained CDC consists of at most eleven induced cycles $C_1', C_2', C_3', C_4, ..., C_{11}$. As the subgraphs G_1', G_2' are disjoint, we merge the cycles that occur in only one subgraph. Note that a cut edge would be weak if we merged two cycles that both contain an end vertex of the same cut edge.

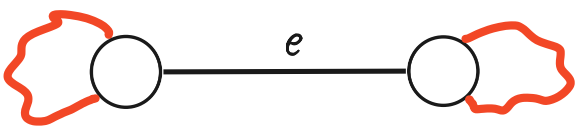


Figure 2.11: An edge e would be weak if we merged the wrong cycles

We want to show that we can merge these cycles such that there is not a weak edge and consequently get an induced 7-CDC. Assume that the original 7-CDCs (of G_1 and

G_2) consisted of exactly seven cycles. As there can be at most three different third cycles passing through the end vertices of the cut edges in the same subgraph, there is a cycle in both subgraphs that does not contain any of the considered vertices, so we can freely merge such cycles with any cycle in the other subgraph and get a bigger induced cycle.

If the third cycles that pass through the vertices v_1, v_2, v_3 and v_4, v_5, v_6 are different from one another, we can pair the cycles such that we get an induced 7-CDC consisting of cycles $C'_1, ...C'_7$, as shown in the figure below. Notice that the cycle C'_7 is the cycle that does not contain any of the vertices $v_1, ..., v_6$.

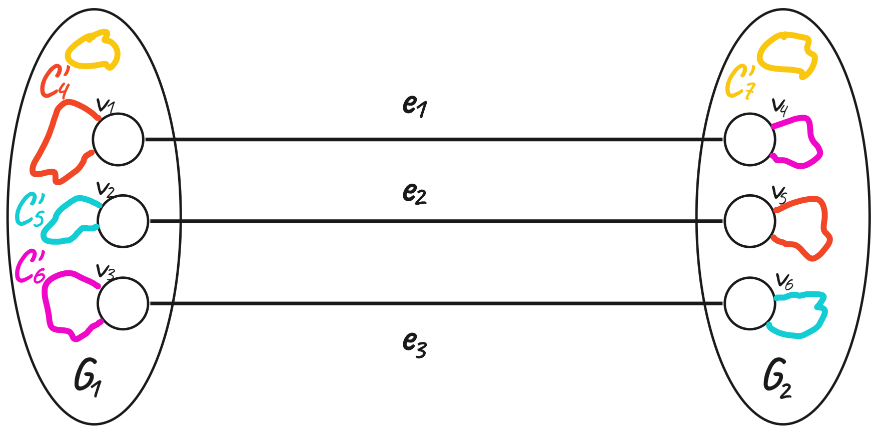


Figure 2.12: One of the possible pairings of the cycles

Notice that if in G_1 were a cycle that passes through exactly two of the vertices v_1, v_2, v_3 as the third cycle, there would be two cycles in G_1 that do not contain any of the vertices v_1, v_2, v_3. In addition, if in G_1 were a cycle that passes through all of the vertices v_1, v_2, v_3 as the third cycle, there would be three cycles in G_1 with such property. The same argument holds for G_2.

Let us consider the case when the CDC consists of a cycle C_4 that passes through all vertices v_1, v_2, v_3, and the third cycles C_8, C_9, C_{10} that pass through v_4, v_5, v_6, respectively, are different from one another.

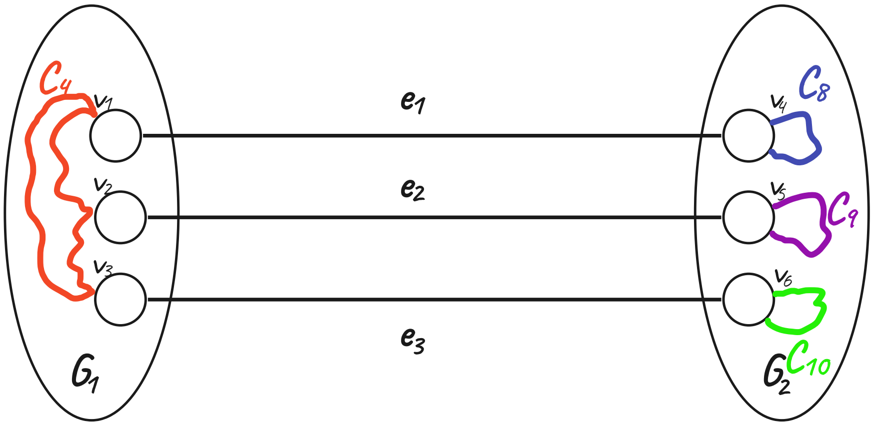


Figure 2.13: The cycles C_4, C_8, C_9, C_{10}

In this case, we can merge these cycles, as shown in the figure below. Notice that if we considered an induced 6-CDC, we would not be able to merge the cycle C_4 such that there is not a weak edge, as there would not be any cycle in G_2 that does not contain any of the vertices v_4, v_5, v_6. Thus, we would not be able to get an induced 6-CDC in general.

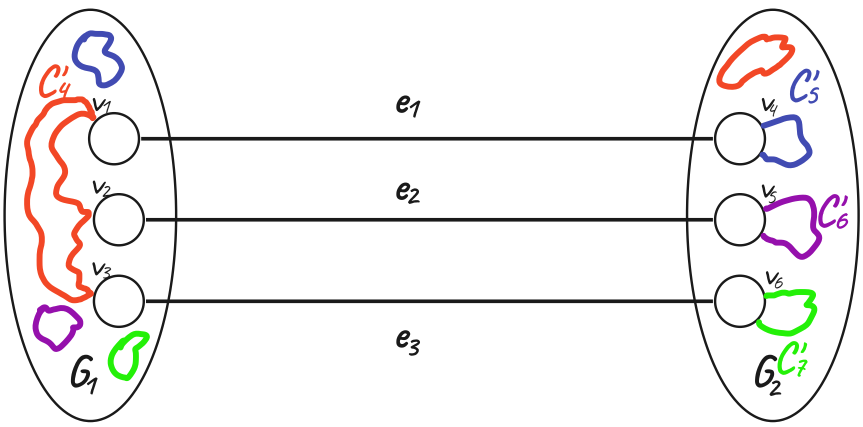


Figure 2.14: The resulting pairing of the cycles

Observe that providing the 7-CDCs of G_1, G_2 consisted of less than seven induced cycles, we would not have to merge some cycles in order to get an induced 7-CDC. Since the fact that a CDC possesses cycles containing more than one of the vertices $v_1, ..., v_6$ as the third cycle implies the existence of more cycles that are independent of the 3-edge cut, the obtained graph has an induced 7-CDC. □

The implication *"If G has a semi-induced 6-CDC (an induced 7-CDC), then G_1 and G_2 have a semi-induced 6-CDC (an induced 7-CDC)."* does not hold true as we found counterexamples with our software.

2.3 Results regarding triangles

Lemma 2.3.1 *Let G be a bridgeless cubic graph with a (semi-)induced cycle double cover $\mathcal{C}$. If G contains a triangle v_1, v_2, v_3, then $\mathcal{C}$ contains a cycle comprising the circuit $v_1v_2v_3v_1$.*

Proof: Let G has a triangle v_1, v_2, v_3. Suppose that $\mathcal{C}$ does not have a cycle comprising the circuit $v_1v_2v_3v_1$. As one readily observes, the CDC $\mathcal{C}$ cannot contain a cycle covering two edges of the triangle since it would result in a weak edge. Thus, a contradiction to the (semi-)induced cycle double cover.

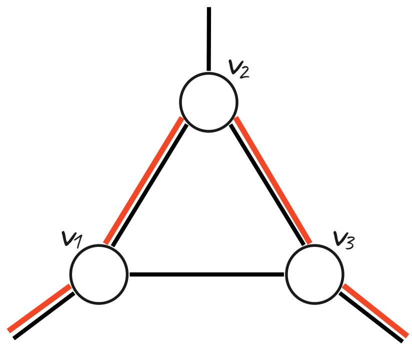


Figure 2.15: The weak edge v_1v_3

As a consequence, $\mathcal{C}$ consists of cycles that cover at most one of the triangle's edges. Without loss of generality, let us consider the edge v_1v_2. It must be covered by two cycles.

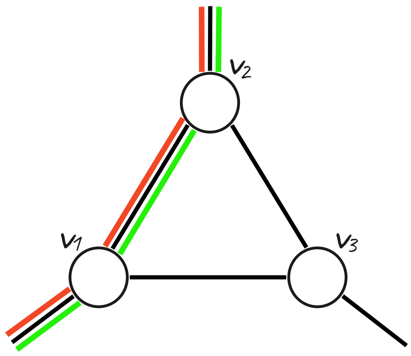


Figure 2.16: Cycle double cover of the edge v_1v_2

Consequently, two of the outgoing edges from the triangle are covered by two cycles. Therefore the edges v_2v_3 and v_1v_3 cannot be covered. Thus, a contradiction to the (semi-)induced cycle double cover. □

Lemma 2.3.2 *Let G be a bridgeless cubic graph and u be a vertex of G. Let G' be a cubic graph obtained by an inflation of the vertex u.*

(a) If G has a semi-induced 6-CDC, then G' also has a semi-induced 6-CDC.

(b) If G has an induced 7-CDC, then G' also has an induced 7-CDC.

Proof: (a) Assume that G has a semi-induced 6-CDC consisting of cycles $C_1, ..., C_6$. Assume that u is a vertex of G and vertices v_1, v_2, v_3 are incident to u. Let the cycles C_1, C_2, C_3 cover the edges outgoing from u.

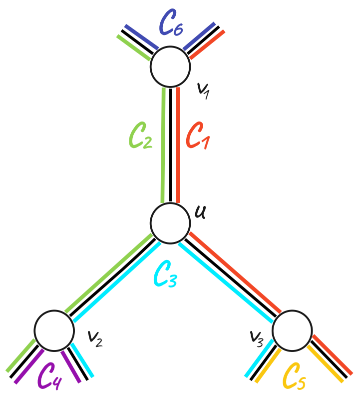


Figure 2.17: A semi-induced 6-CDC of G

We obtain the graph G' by an inflation of the vertex u. The inflation results in a triangle u_1, u_2, u_3. We need to cover all edges of the triangle, u_1u_2, u_2u_3, u_3u_1, as well as the outgoing edges from the triangle, u_1v_1, u_2v_2, u_3v_3, using the original 6-CDC. We can cover these edges, as shown in the figure below.

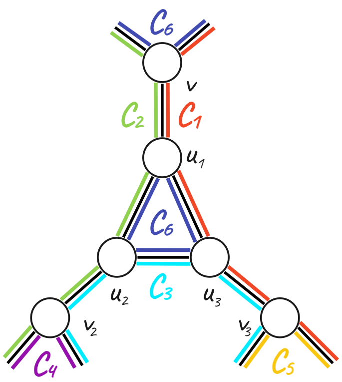


Figure 2.18: A semi-induced 6-CDC of G'

Observe that we extended C_1, C_2, C_3 without getting a weak edge. We also added a new chordless circuit $u_1u_2u_3u_1$ that we merged with the cycle C_6 as they do not share any vertex. Hence, G' has a semi-induced 6-CDC.

(b) Assume that G has an induced 7-CDC consisting of cycles $C_1, ..., C_7$. The argument is the same as in (a), but instead of merging the circuit $u_1u_2u_3u_1$ with the cycle C_6, we merge the circuit with the cycle C_7. Hence, G' has an induced 7-CDC.

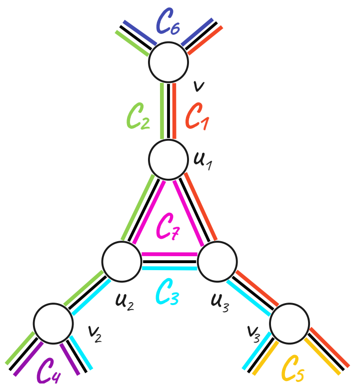


Figure 2.19: Induced 7-CDC of G'

□

Definition 2.3.3 *The E-inflation of an edge uv is the removal of the edge uv and the addition of vertices $w_1, ...w_4$ along with edges $uw_1, w_1w_2, w_1w_3, w_2w_3, w_2w_4, w_3w_4, w_4v$.*

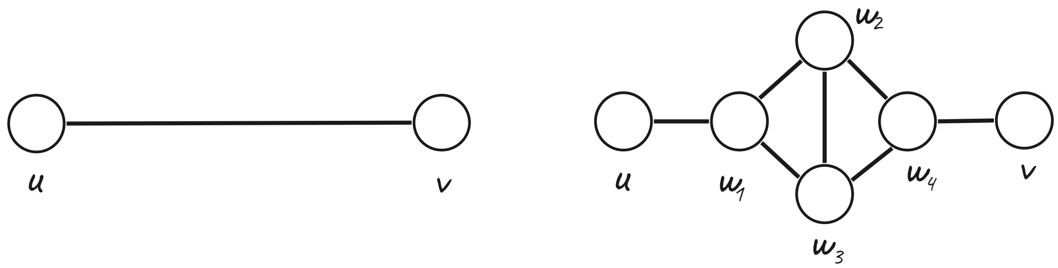


Figure 2.20: The E-inflation of an edge uv

Observe that if a graph is cubic and uv is an edge of the graph, then the graph obtained by the E-inflation of the edge uv is cubic as well.

Lemma 2.3.4 *Let G be a bridgeless cubic graph with an edge uv and let G' be the graph obtained by the E-inflation of the edge uv.*

(a) If G has a semi-induced 6-CDC, then G' also has a semi-induced 6-CDC.

(b) If G has an induced 7-CDC, then G' also has an induced 7-CDC.

Proof: (a) Assume that G has a semi-induced 6-CDC consisting of cycles $C_1, ..., C_6$. Assume that an edge uv is covered by the cycles C_1, C_2, and the third cycles passing through the vertices u and v are C_3 and C_4, respectively.

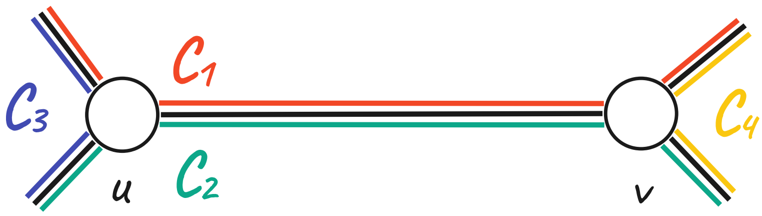


Figure 2.21: The cycle double cover of the edge uv

We obtain the graph G' by the E-inflation of the edge uv. We can cover the new edges with $C_1, ...C_6$, as shown in the figure below. As a consequence, the obtained graph G' has a semi-induced 6-CDC.

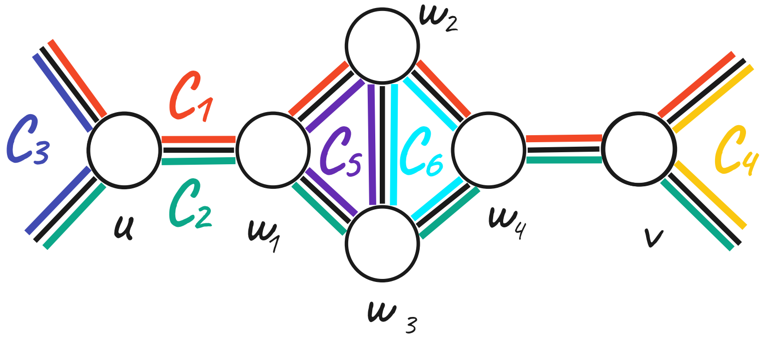


Figure 2.22: The cycle double cover of the new edges

(b) Assume that G has an induced 7-CDC consisting of cycles $C_1, ..., C_7$. The same argument holds as in (a), as there is not a weak edge, so the obtained graph G' has an induced 7-CDC. $\square$

Chapter 3

Our Conjecture

In this chapter, we state our own conjecture about bridgeless cubic graphs that are cyclically 4-edge-connected with girth at least 5 and prove our conjecture on some infinite graph families.

We used our software to test bridgeless cubic graphs with various properties for semi-induced and induced cycle double covers, and subsequently, we conjecture:

Conjecture 3.0.1 *Every bridgeless cyclically 4-edge-connected cubic graph with girth at least 5:*

(a) has a semi-induced 6-CDC.

(b) has an induced 7-CDC.

We verified the conjecture with our software on all bridgeless cyclically 4-edge-connected cubic graphs with girth at least 5 up to 24 vertices and on all nontrivial snarks up to 28 vertices.

We will use the following proposition in some proofs of our conjecture. It allows us to prove both parts of our conjecture at once.

Proposition 3.0.2 *If a graph has an induced 6-CDC, then it has a semi-induced 6-CDC and an induced 7-CDC.*

We will prove our conjecture on some infinite families of snarks, which are derivated by the junction of multipoles that we define below in addition to related terms.

Definition 3.0.3 *A dangling edge is an edge with only one end vertex.*

Definition 3.0.4 *An isolated edge is an edge without end vertices.*

The dangling and isolated edges are collectively called *semiedges*.

Definition 3.0.5 *A multipole $M = (V, E, \mathcal{E})$ is a graph comprising a finite set of vertices V, a set of edges E, and a set of semiedges $\mathcal{E}$.*

In the following sections, we present the infinite families of graphs that we tried to prove our conjecture on. Some methods that we used to find the presented cycle double covers are described in the chapter Implementation.

3.1 Isaacs snarks

Isaacs snarks, also known as the flower snarks, form an infinite family of snarks discovered by an American game theorist Rufus Isaacs in 1975 [5].

Definition 3.1.1 *The $(3, 3)$-pole Y is a multipole consisting of 4 vertices, 3 edges, and 6 dangling edges, as shown in the figure below:*

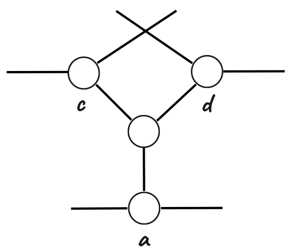


Figure 3.1: The $(3, 3)$-pole Y

We denote the vertices incident to the dangling edges of the i^{th} copy of the $(3, 3)$-pole Y as a_i, c_i, d_i. The Isaacs snark J_n (for an odd integer $n \geq 3$) can be derived by the cyclic connection of n copies of the $(3, 3)$-pole Y via:

- The addition of the edges $a_i a_{i+1}$ for all $1 \leq i < n$, along with the edge $a_1 a_n$.
- The addition of the edges $c_i d_{i+1}$ for all $1 \leq i < n$, along with the edge $c_1 d_n$.
- The addition of the edges $d_i c_{i+1}$ for all $1 \leq i < n$, along with the edge $d_1 c_n$.

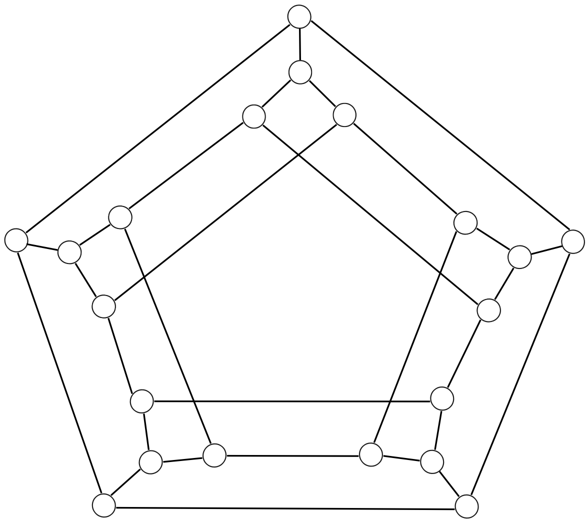


Figure 3.2: J_5

Theorem 3.1.2 *The graph J_n for an odd integer $n \geq 3$ has:*

(a) a semi-induced 6-CDC.

(b) an induced 7-CDC.

Proof: (a) We split the proof into three parts. Firstly, we prove the theorem for $n = 3$. Then we prove the theorem for $n = 5 + 4k$ for a non-negative integer k. Lastly, we prove the theorem for $n = 7 + 4k$ for a non-negative integer k. An induced 6-CDC of J_3 consisting of cycles $C_1, ..., C_6$ is depicted in the figure below.

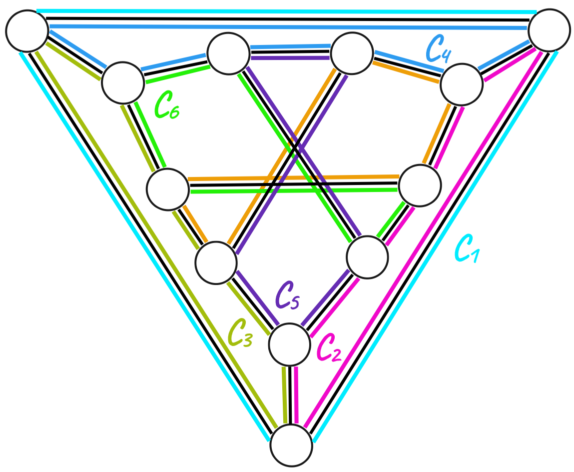


Figure 3.3: An induced 6-CDC of J_3

Let us consider J_{5+4k}. We construct J_{5+4k} from J_5 by the addition of $4k$ copies of the $(3,3)$-pole Y. Assume we have already inserted $(k-1)$ groups of four $(3,3)$-poles Y into J_5, and we want to insert the k^{th} group. We insert the k^{th} group for a positive integer k accordingly:

- The removal of the edges $a_1a_{2+4(k-1)}, c_1d_{2+4(k-1)}$ and $d_1c_{2+4(k-1)}$ if $k \geq 2$.
  (we remove these edges from the obtained graph)

- The removal of the edges a_1a_2, c_1d_2 and d_1c_2 if $k = 1$.
  (we remove these edges from J_5)

- The removal of the dangling edges.
  (we remove these edges from the group of four copies of the $(3, 3)$-pole Y)

- The addition of the edges a_1a_{2+4k}, c_1d_{2+4k} and d_1c_{2+4k}.

- The addition of the edges $a_{2+4(k-1)}a_{5+4k}, c_{2+4(k-1)}d_{5+4k}$ and $d_{2+4(k-1)}c_{5+4k}$.

In the figures below, we depict an induced 6-CDC of J_5 consisting of cycles $C_1, ..., C_6$ that are, in fact, circuits, and the induced 6-CDC that we will use to extend the cycles $C_1, ..., C_6$ when adding the k^{th} group of four copies of the $(3, 3)$-pole Y. Note that the cycles $C_1, ..., C_6$ are still circuits after the extension.

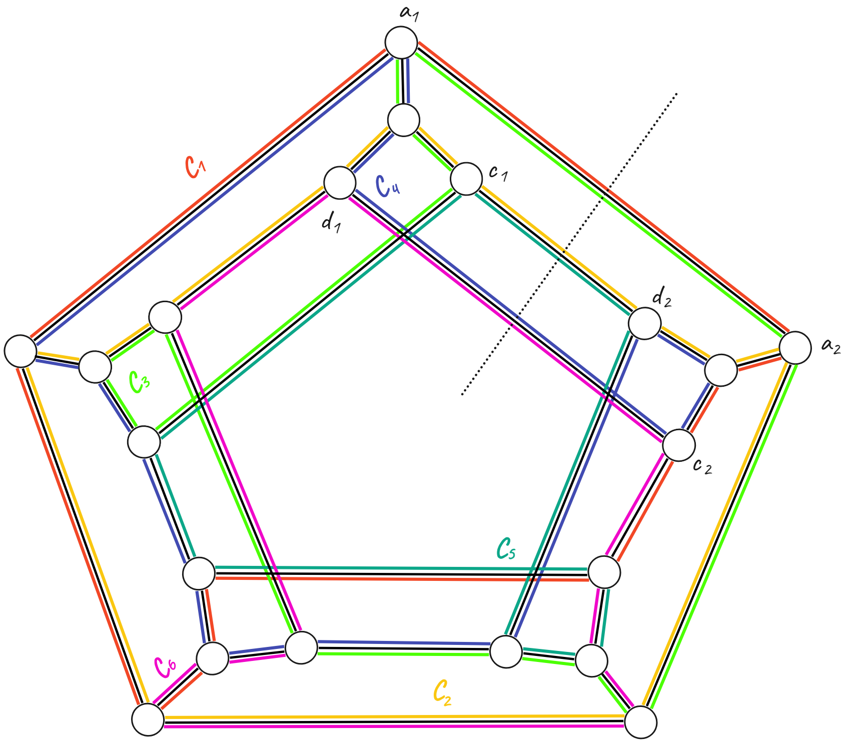


Figure 3.4: J_5 with an induced 6-CDC

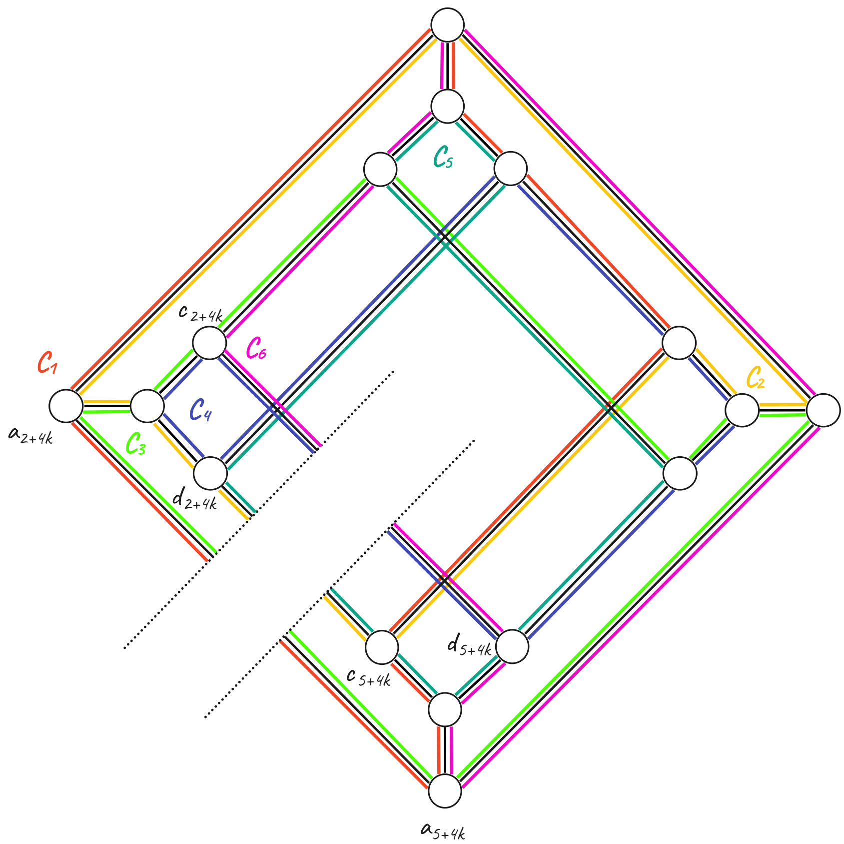


Figure 3.5: The induced 6-CDC of the group of four copies of the (3, 3)-pole Y

In the figure below, we depict the obtained induced 6-CDC of J_9. Note that the vertices of the original J_5 are denoted by the black color.

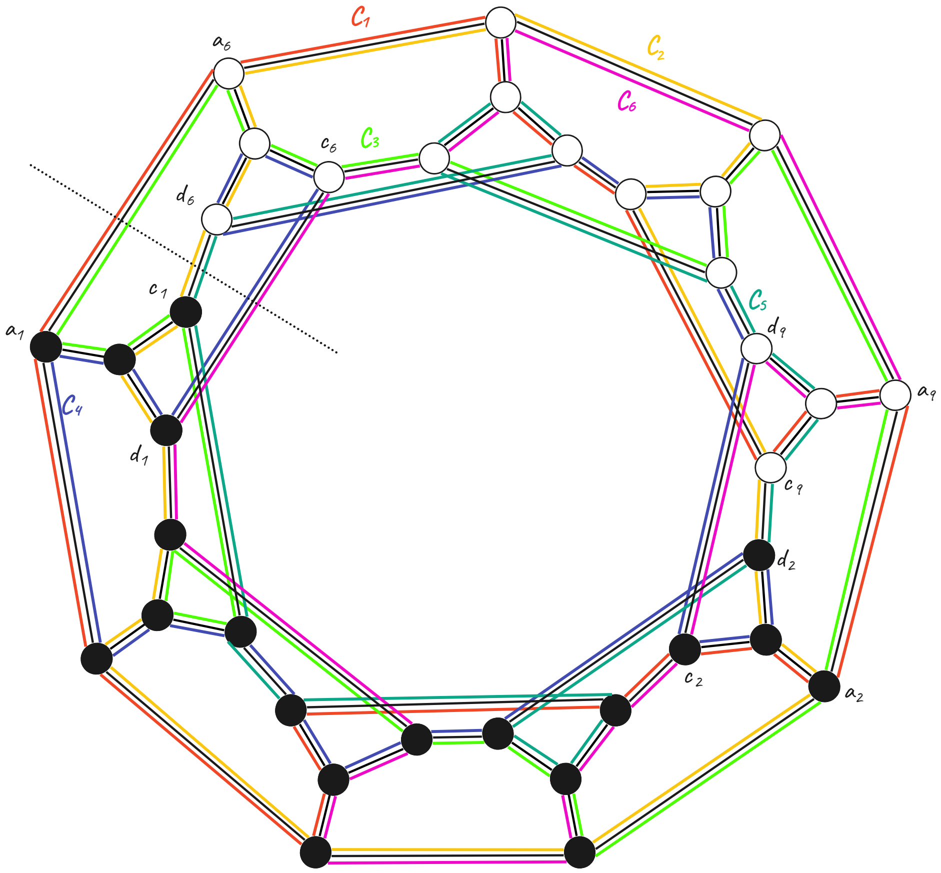


Figure 3.6: The obtained induced 6-CDC of J_9

Observe that the cycles extend correctly because the cycles passing through the edges that we remove correspond to the cycles passing through the edges that we add. As the cycles are, in fact, induced circuits, we proved that J_{5+4k} has an induced 6-CDC. Hence, we proved the theorem for J_{5+4k} for a non-negative integer k (See Proposition 3.0.2).

Analogously, we prove the theorem for J_{7+4k} that we construct from J_7 by the addition of $4k$ copies of the $(3,3)$-pole Y, similarly to the previous case. In the figures below, we depict an induced 6-CDC of J_7 and the induced 6-CDC that we use to extend the induced 6-CDC of J_7. Notice that the cycles are, in fact, induced circuits, even after the extension.

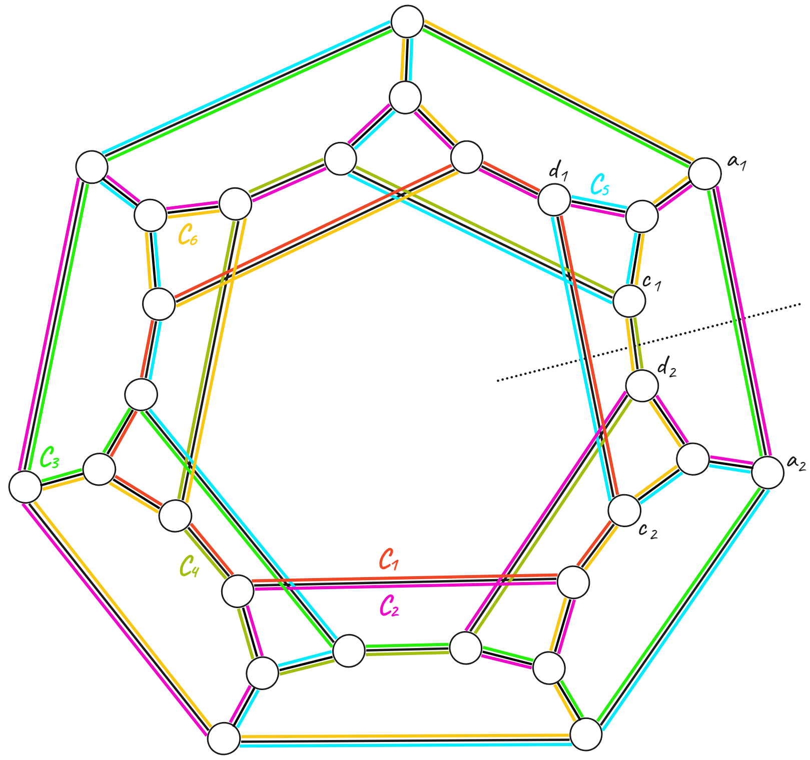


Figure 3.7: J_7 with an induced 6-CDC

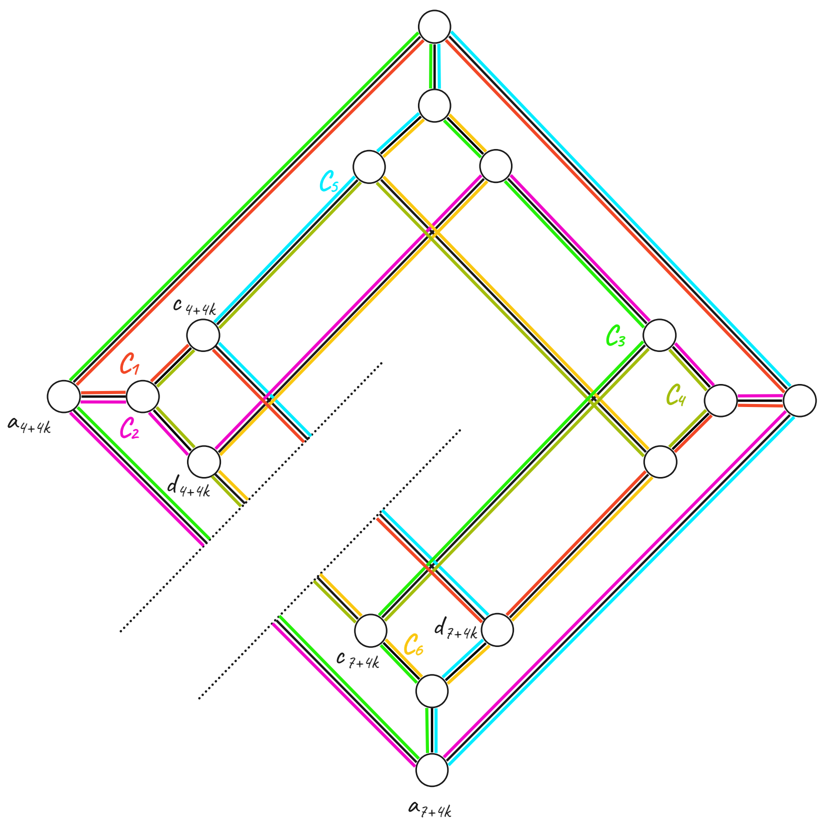


Figure 3.8: The induced 6-CDC of the group of four copies of the $(3,3)$-pole Y

$\square$

3.2 Generalized Blanuša snarks

Generalized Blanuša snarks form an infinite family of snarks, which was introduced by John J. Watkins in 1989 [3].

Definition 3.2.1 *The* $(2, 2)$*-pole* B *is a multipole consisting of 8 vertices, 10 edges and 4 dangling edges, as shown in the figure below:*

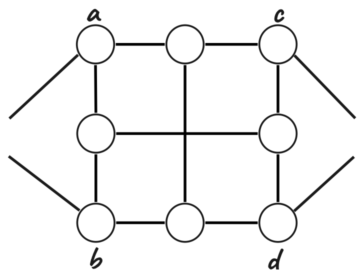


Figure 3.9: The $(2, 2)$-pole B

The Blanuša snark B_1 is the Petersen graph [8].

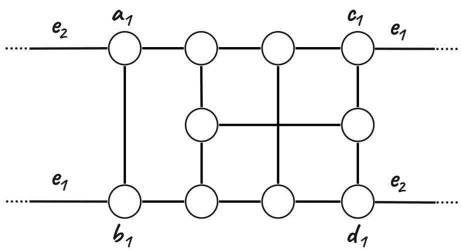


Figure 3.10: The Petersen graph

Let us denote the vertices incident to the dangling edges of the i^{th} copy of the $(2, 2)$-pole B as $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$. The Blanuša snark B_n (for a positive integer $n \geq 2$) is derived from B_1 as follows:

- The removal of the edges a_1d_1, b_1c_1.
  (we remove these edges from B_1*)*

- The removal of the dangling edges.
  (we remove these edges from $(2, 2)$*-poles* B*)*

- The addition of the edges c_ib_{i+1} for all $1 \leq i < n$, along with the edge c_nb_1.

- The addition of the edges d_ia_{i+1} for all $1 \leq i < n$, along with the edge d_na_1.

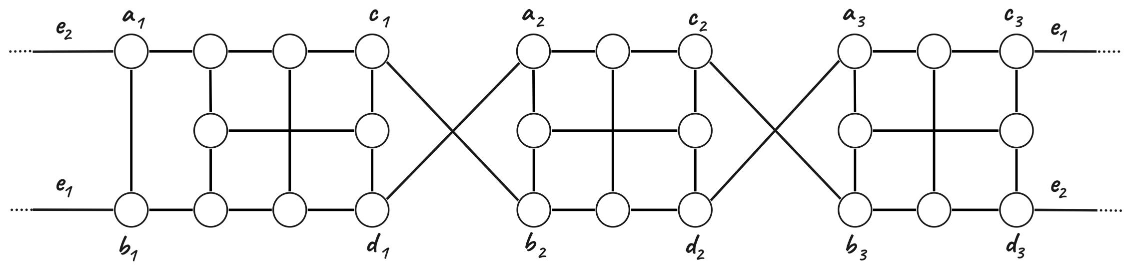


Figure 3.11: B_3

Theorem 3.2.2 *The graph B_n for a positive integer n has:*

(a) a semi-induced 6-CDC.

(b) an induced 7-CDC.

We split the proof into three parts. Firstly, we prove the theorem for $n = 1$. Then we prove the theorem for $n = 2 + 2k$ for a non-negative integer k. Lastly, we prove the theorem for $n = 3 + 2k$ for a non-negative integer k. In the figure below, we depict an induced 6-CDC of B_1 consisting of cycles $C_1, ..., C_6$ that are, in fact, circuits, and by Proposition 3.0.2, we proved the theorem for B_1.

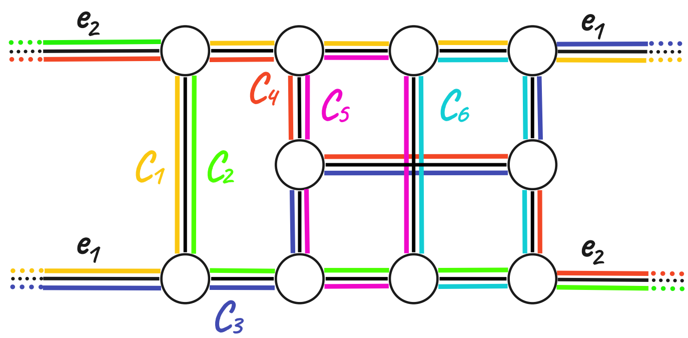


Figure 3.12: An induced 6-CDC of B_1

Now we prove the theorem for $n = 2+2k$ for a non-negative integer k. We construct B_{2+2k} from B_2 by the addition of $2k$ copies of $(2, 2)$-poles B. Assume we have already inserted $(k - 1)$ groups of two $(2, 2)$-poles B into B_2, and we want to insert the k^{th} group. We insert it as follows:

- The removal of the edges c_1b_2, d_1a_2 if $k = 1$.
  (we remove these edges from B_2)

- The removal of the edges $c_{2k}b_2, d_{2k}a_2$ if $k \geq 2$.
  (we remove these edges from the obtained graph)

- The removal of the dangling edges.

- The addition of the edges c_1b_3, d_1a_3 if $k = 1$.
- The addition of the edges $c_{2k}b_{2k+1}, d_{2k}a_{2k+1}$ if $k \geq 2$.
- The addition of the edges $c_{2k+2}b_2, d_{2k+2}a_2$.

In the figures below, we depict an induced 6-CDC of B_2, consisting of cycles $C_1, ..., C_6$, and the induced 6-CDC that we will use to extend the cycles when adding the k^{th} group of two copies of the $(2, 2)$-pole B.

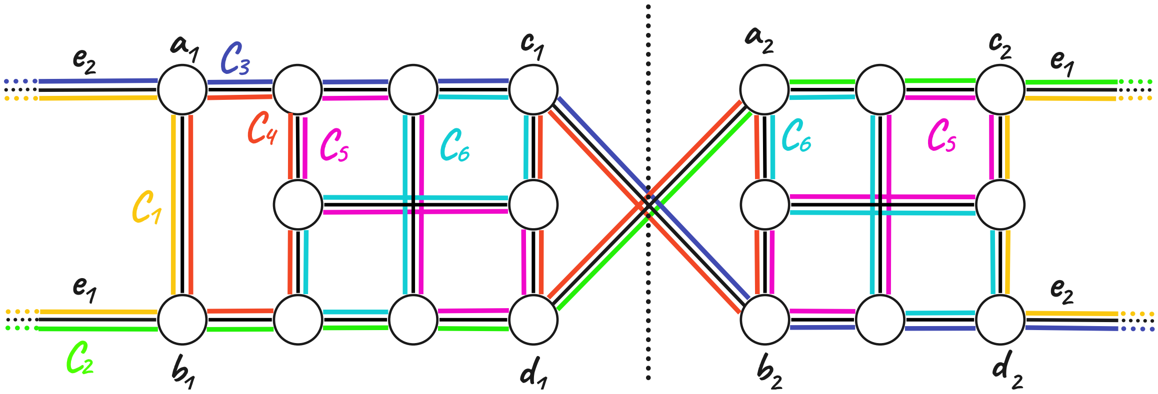


Figure 3.13: An induced 6-CDC of B_2

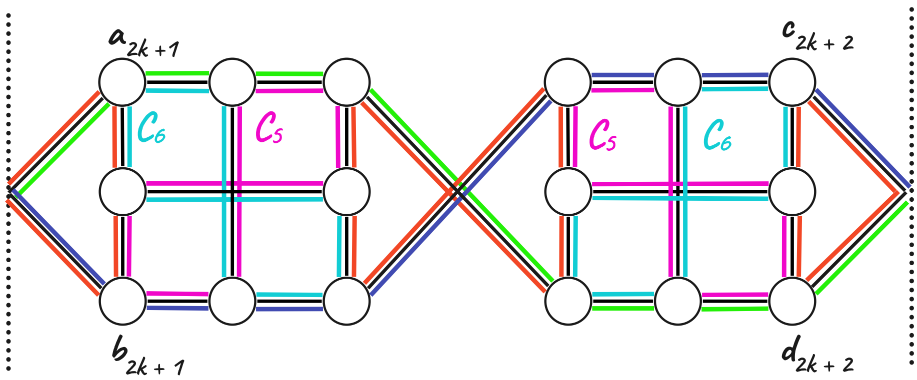


Figure 3.14: The induced 6-CDC of the group of two copies of the $(2, 2)$-pole B

In the figure below, we depict the obtained induced 6-CDC of B_4. Note that the vertices of the original B_2 are denoted by the black color.

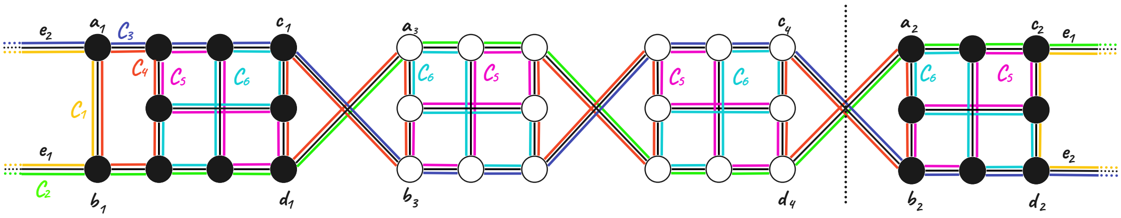


Figure 3.15: The obtained induced 6-CDC of B_4

Note that the cycles extend correctly as the cycles passing through the edges that we remove correspond to the cycles passing through the edges that we add. The 6-CDC remains induced because the third cycles passing through the end vertices of the edges that we remove correspond to the third cycles passing through the end vertices of the edges that we add. We proved that B_{2+2k} has an induced 6-CDC, and as a consequence, we proved the theorem for B_{2+2k} for a non-negative integer k (See Proposition 3.0.2).

Analogously, we prove the theorem for B_{3+2k} that we likewise construct from B_3 via the addition of $2k$ copies of the $(2,2)$-pole B. In the figures below, we depict an induced 6-CDC of B_3 and the induced 6-CDC that we use to extend the induced 6-CDC of B_3.

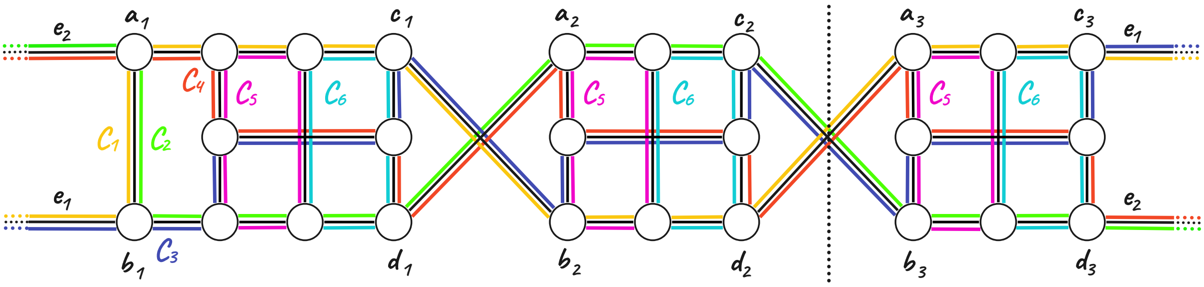


Figure 3.16: An induced 6-CDC of B_3

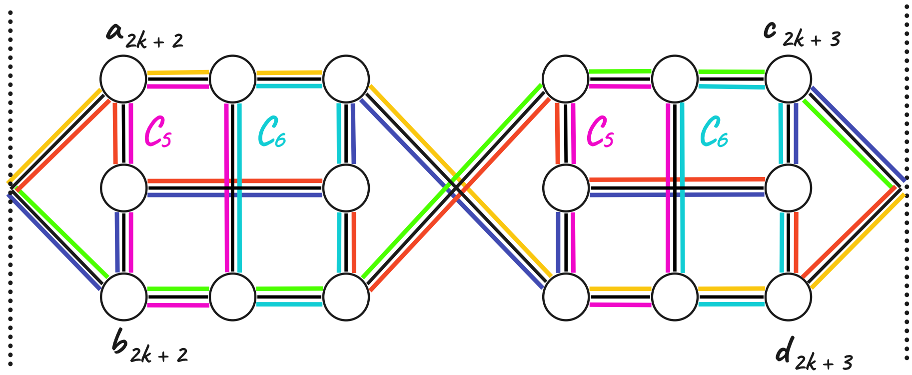


Figure 3.17: The induced 6-CDC of the group of two copies of the $(2,2)$-pole B

The 6-CDC stays induced for the same reason as in the previous case. □

3.3 Loupekine snarks

Loupekine snarks form an infinite family of snarks named after F. Loupekine [7].

Definition 3.3.1 *The* $(3,3)$*-pole* L *is a multipole consisting of 8 vertices, 8 edges and 6 dangling edges, as shown in the figure below:*

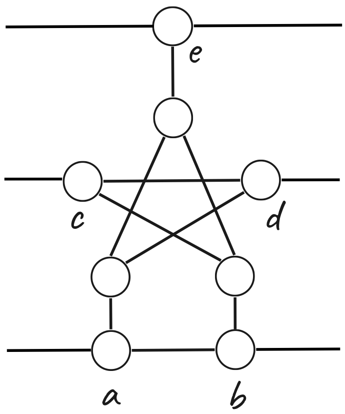


Figure 3.18: The $(3,3)$-pole L

Let us denote the vertices incident to the dangling edges of the i^{th} copy of the $(3,3)$-pole L as a_i, b_i, c_i, d_i, e_i. The Loupekine snark L_n (for a positive odd integer $n \geq 3$) is derived by the cyclic connection of n copies of the $(3,3)$-pole L by:

- The addition of the edges $b_i a_{i+1}$ for all $1 \leq i < n$, along with the edge $a_1 b_n$.
- The addition of the edges $d_i c_{i+1}$ for all $1 \leq i < n$, along with the edge $c_1 d_n$.
- The addition of the edges $e_i e_{i+1}$ for all $1 \leq i < n$, along with the edge $e_1 e_n$.

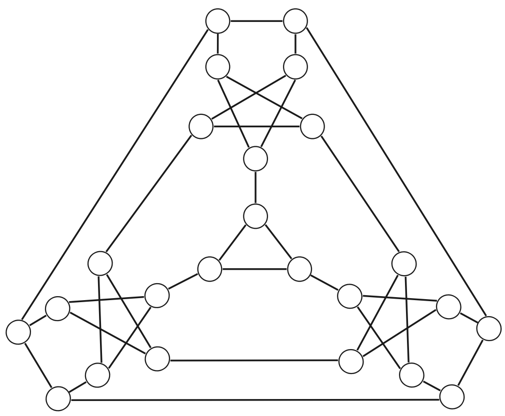


Figure 3.19: L_3

Theorem 3.3.2 *The graph L_n for a positive odd integer $n \geq 3$ has a semi-induced 6-CDC.*

Proof: We construct L_{3+2k} from L_3 by the addition of $2k$ copies of the $(3,3)$-pole L. Assume we have already inserted $(k-1)$ groups of two $(3,3)$-poles L into L_3, and we want to insert the k^{th} group. We insert it as follows:

- The removal of the edges a_1b_2, c_1d_2, e_1e_2 if $k = 1$.
  (we remove these edges from J_3)
- The removal of the edges $a_1b_{2(k-1)+2}, c_1d_{2(k-1)+2}, e_1e_{2(k-1)+2}$ if $k \geq 2$.
  (we remove these edges from the obtained graph)
- The removal of the dangling edges.
  (we remove these edges from the group of two copies of the $(3,3)$-pole L)
- The addition of the edges $b_{2(k-1)+2}a_{2k+1}, d_{2(k-1)+2}c_{2k+1}, e_{2(k-1)+2}e_{2k+1}$.
- The addition of the edges $a_1b_{2k+2}, c_1d_{2k+2}, e_1e_{2k+2}$.

In the figures below, we depict a semi-induced 6-CDC of L_3 consisting of cycles $C_1, ..., C_6$, and the semi-induced 6-CDC that we will use to extend the cycles when adding the k^{th} group of two copies of the $(3,3)$-pole L.

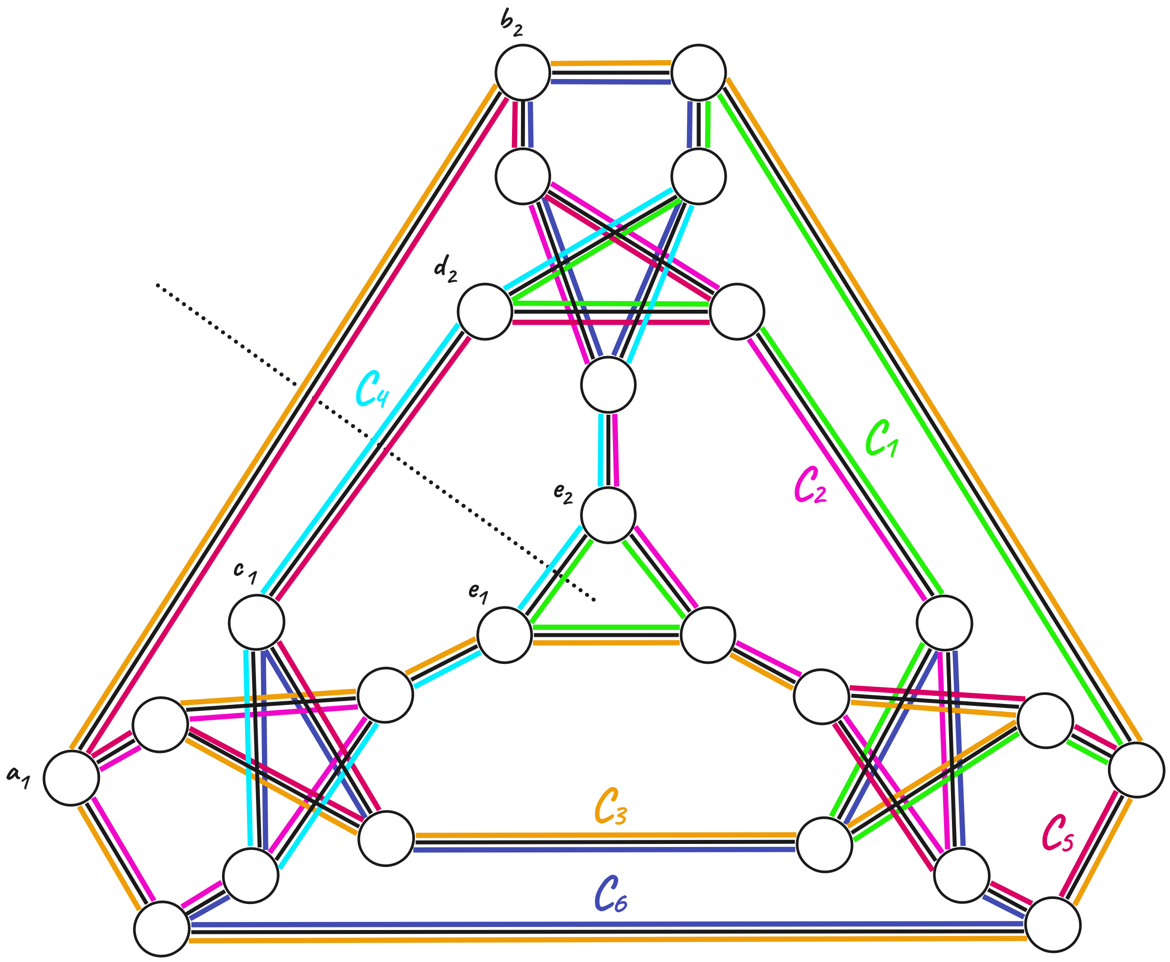


Figure 3.20: A semi-induced 6-CDC of L_3

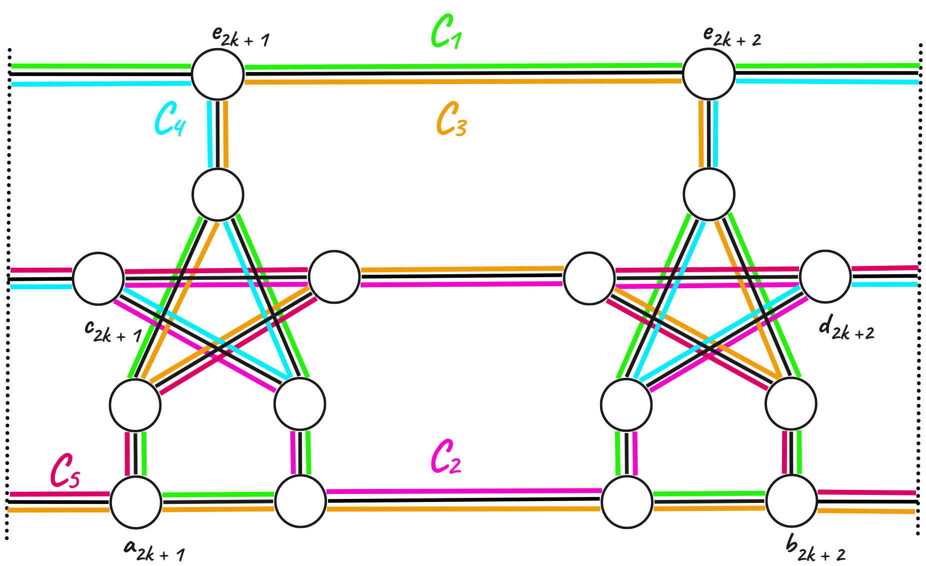


Figure 3.21: The semi-induced 6-CDC of the group of two copies of the $(3, 3)$-pole L

In the figure below, we depict the obtained semi-induced 6-CDC of L_5. Note that the vertices of the original L_3 are denoted by the black color.

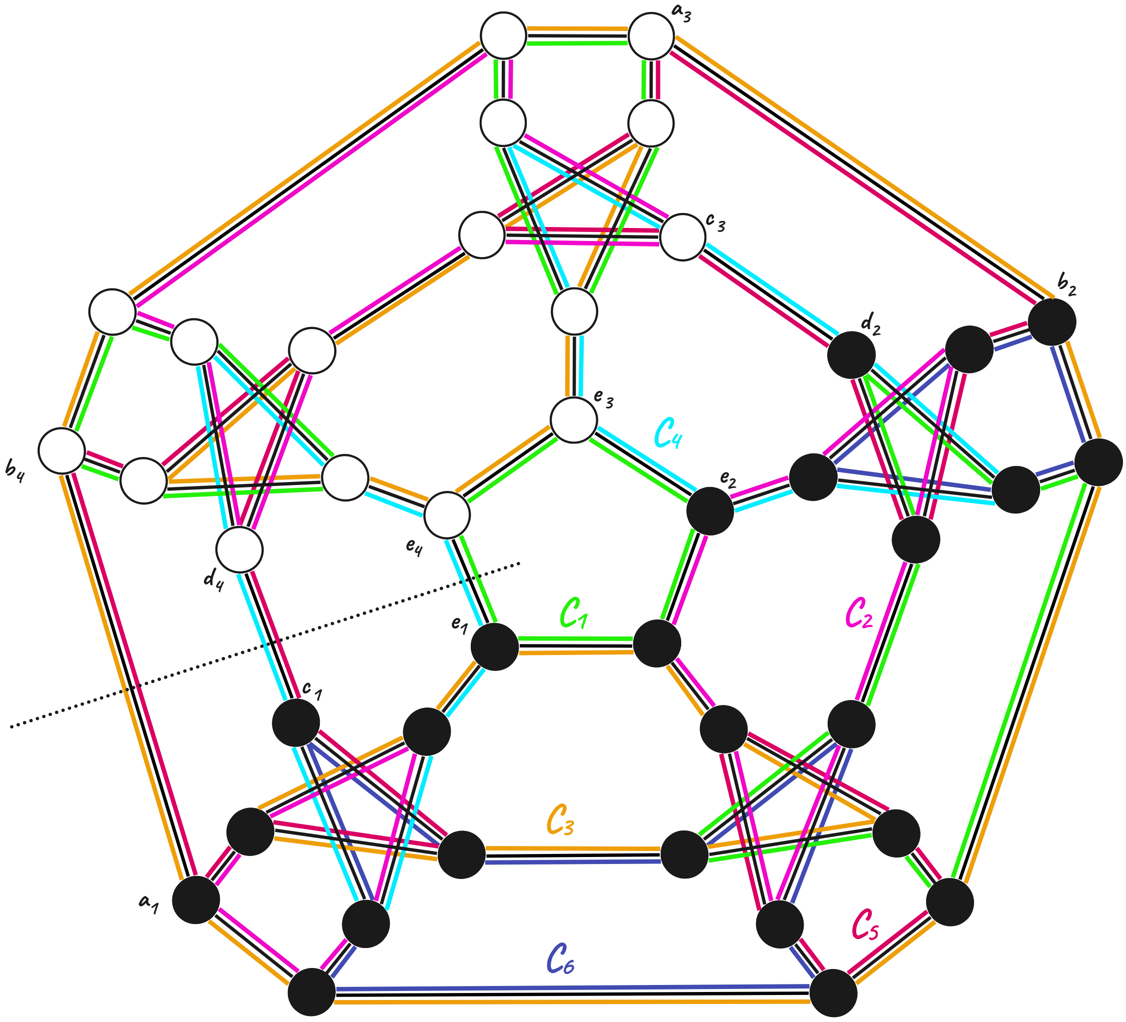


Figure 3.22: The obtained semi-induced 6-CDC of L_5

Note that the cycles extend correctly as the cycles passing through the edges that we remove correspond to the cycles passing through the edges that we add. The cycles remain semi-induced because the 6-CDC of the group of two $(3, 3)$-poles L is semi-induced. Hence, we proved that Loupekine snarks have a semi-induced 6-CDC. $\square$

Theorem 3.3.3 *The graph L_n for $n = 3 + 4k$ for a non-negative integer k has an induced 7-CDC.*

We construct L_{3+4k} from L_3 by the addition of $4k$ copies of the $(3, 3)$-pole L. Assume we have already inserted $(k-1)$ groups of four $(3, 3)$-poles L into L_3, and we want to insert the k^{th} group. We insert it followingly:

- The removal of the edges b_1a_2, d_1c_2, e_1e_2 if $k = 1$.
  (we remove these edges from J_3)
- The removal of the edges $a_2b_{4(k-1)}, c_2d_{4(k-1)}, e_2e_{4(k-1)}$ if $k \geq 2$.
  (we remove these edges from the obtained graph)
- The removal of the dangling edges.
  (we remove these edges from the group of four copies of $(3, 3)$-pole L)
- The addition of the edges $b_1a_{3+4k}, d_1c_{3+4k}, e_1e_{3+4k}$ if $k = 1$.
- The addition of the edges $b_{4(k-1)}a_{3+4k}, d_{4(k-1)}c_{3+4k}, e_{4(k-1)}e_{3+4k}$ if $k \geq 2$.
- The addition of the edges $a_2b_{4k}, c_2d_{4k}, e_2e_{4k}$.

In the figures below, we depict an induced 7-CDC of L_3 as well as the induced 7-CDC that we use to extend the cycles when adding the k^{th} group of four copies of the $(3, 3)$-pole L. Note that the obtained 7-CDC is induced because the third cycles of the end vertices of the added edges remain different. In L_7, the third cycles passing through the end vertices of:

- the edge b_1a_7 are C_1, C_2.
- the edge d_1c_7 are C_3, C_5.
- the edge e_1e_7 are C_2, C_3.
- the edge a_2b_4 are C_3, C_4.
- the edge c_2d_4 are C_4, C_5.
- the edge e_2e_4 are C_1, C_2.

In L_{7+4k}, the third cycles passing through the end vertices of:

- the edges $a_2b_{4k}, c_2d_{4k}, e_2e_{4k}$ are the same as the ones passing through the end vertices of the edges a_2b_4, c_2d_4, e_2e_4, respectively (in L_7).
- the edges $a_{3+4k}b_{4(k-1)}, c_{3+4k}d_{4(k-1)}, e_{3+4k}e_{4(k-1)}$ are the same as the ones passing through the end vertices of the edges $a_{3+4k}b_{4k}, c_{3+4k}d_{4k}, e_{3+4k}e_{4k}$, respectively (in the k^{th} added group of four $(3, 3)$-poles L).

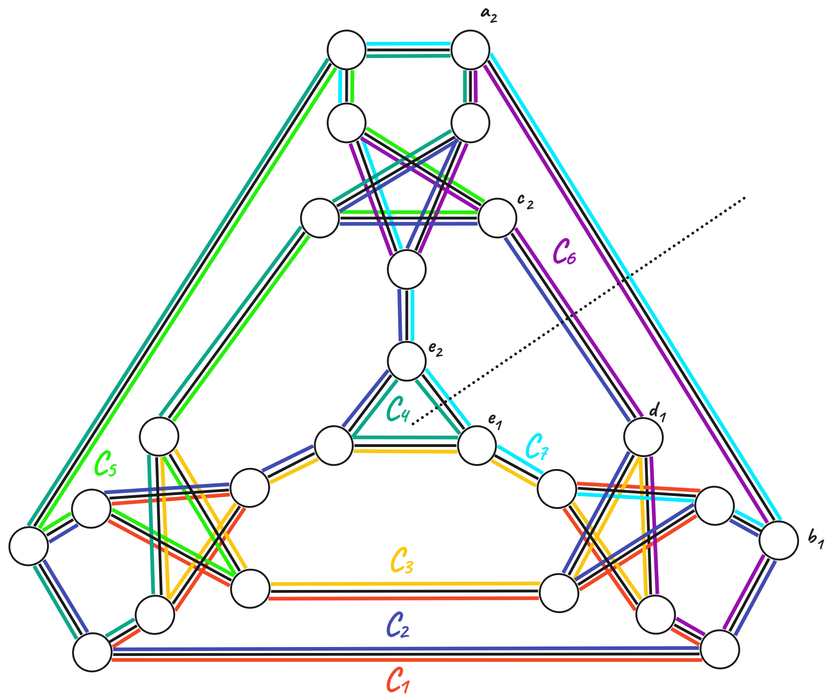


Figure 3.23: An induced 7-CDC of L_3

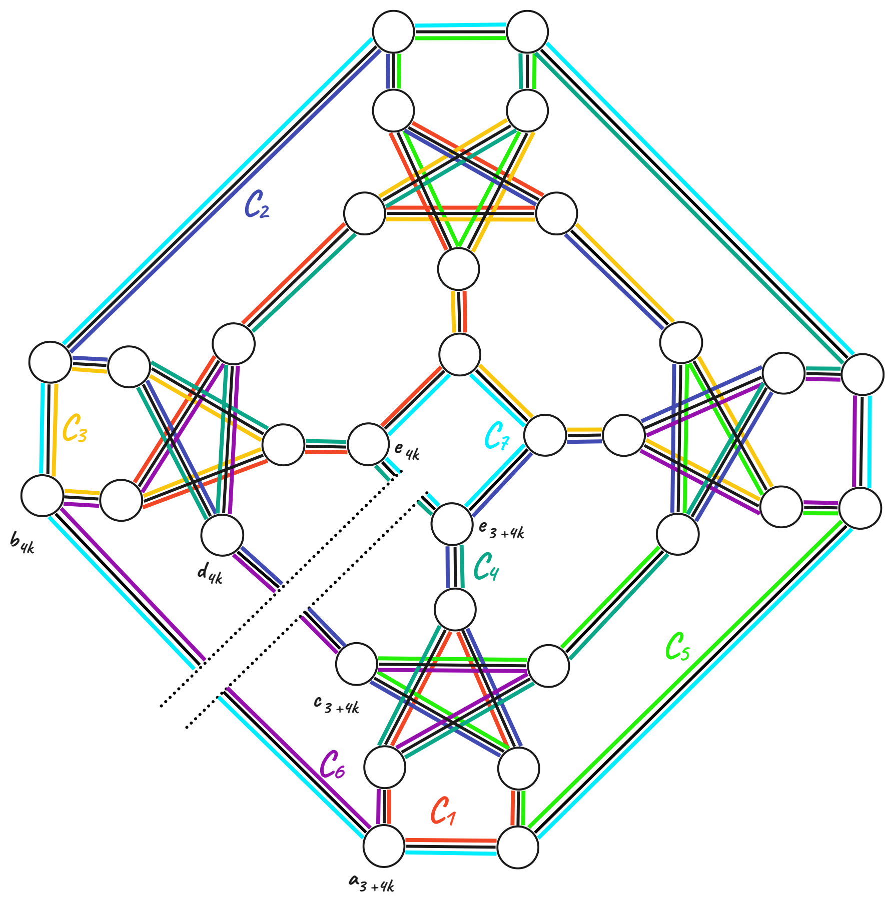


Figure 3.24: The induced 7-CDC of the group of four copies of $(3, 3)$-pole L

In the figure below, we depict the obtained induced 7-CDC of L_7. The vertices of the original L_3 are denoted by the black color as usual.

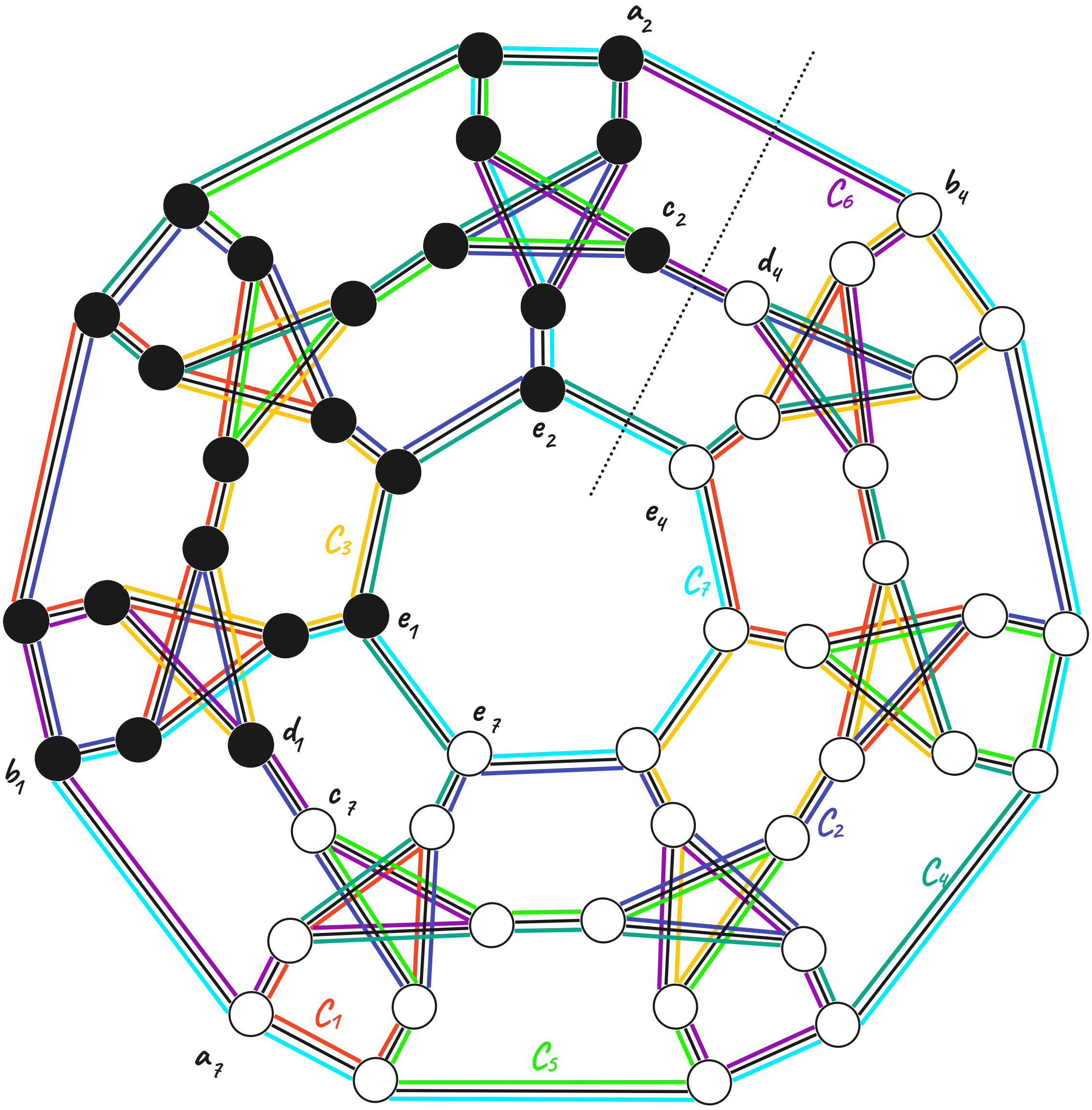


Figure 3.25: The obtained induced 7-CDC of L_7

The cycles extend correctly since the cycles passing through the edges that we remove correspond to the cycles passing through the edges that we add. The 7-CDC of the obtained graphs remains induced as there are no weak edges acquired when extending the cycles. □

We were not able to prove our theorem for J_{5+4k} for a non-negative integer k. Therefore, we conjecture:

Conjecture 3.3.4 *The graph J_{5+4k} for a non-negative integer k has an induced 7-CDC.*

Chapter 4

Implementation

This chapter is dedicated to the implementation of our software we used for verification of our conjecture. We describe how we generated graphs, how we represented a graph in our software, the algorithm we used to find cycle double covers, as well as details about the way we found the CDCs that we used in proofs of our conjecture.

4.1 Generation of graphs

To generate connected cubic graphs, we used the software called Genreg [6]. We generated all cubic graphs up to 16 vertices as well as all cubic graphs with girth at least 5 up to 24 vertices. The format of the generated graphs is shown below.

```
6
3 4 5
2 3 4
1 3 5
0 1 2
0 1 5
0 2 4

2 3 4
2 3 4
0 1 5
0 1 5
0 1 5
2 3 4
```

Figure 4.1: The cubic graphs on 6 vertices.

The number in the first row represents the number of vertices of all graphs. Other rows represent an adjacency list. Graphs are separated by an empty line.

4.2 Graph representation

We implemented our software in Java. For the representation of graphs we used these classes:

- `Vertex` - it contains the number of the vertex and a list of instances of the class `Vertex` representing the adjacent vertices.
- `Edge` - it contains two instances of the class `Vertex` representing the end vertices.
- `Circuit` - it contains a set of instances of the class `Edge` as well as a set of instances of the class `Vertex` that form the circuit. The class provides a method to determine whether the circuit is induced.
- `Graph` - it contains a list of instances of the class `Vertex`, representing the vertices of the graph, and a set of instances of the class `Edge` representing the edges of the graph. It provides methods to determine whether the graph has a bridge, a 2-edge cut, or a nontrivial 3-edge cut. The class also provides a method to find all circuits of the graph.
- `Cycle` - it contains a set of instances of the class `Circuit` that form the cycle. It provides a method to determine whether the cycle is induced.
- `CycleDoubleCover` - it contains a set of instances of the class `Cycle` representing the cycles that form the CDC, as well as a set of instances of the class `Edge` representing the edges of the CDC. It provides a method to find all strong edges in the cycle double cover.

4.3 Algorithm

We implemented an algorithm for finding cycle double covers of a graph. We firstly generated all circuits of the graph, and then we found all cycles of the graphs by merging disjoint circuits into cycles. The basic idea of our algorithm is that it adds a new cycle to a list of cycles if and only if the set of the edges of the newly added cycle and the set of the edges that are already covered twice are disjoint. We find a cycle double cover if the number of edges of the graph equals the number of edges covered twice.

- `cycles` - a list of all cycles of a graph.

- `foundCDCs` - a list of found cycle double covers.
- `any` - a boolean. True if we want to determine whether the graph has any cycle double cover rather than finding all of them (useful to determine whether a CDC with selected properties exists).
- `StopRecursionException` - an exception that is thrown in order to stop the computation immediately.
- `processTwoTimesUsedEdges()` - a function that determines the edges that are already covered by two cycles.

The recursive function `findCycleDoubleCovers()` takes these three parameters:

- `data` - a list of cycles that represents an actual combination of cycles.
- `start` - an integer that represents the position that are new cycles chosen from.
- `twoTimesUsedEdges` - a set of edges that are already covered by two cycles in the actual combination.

The basic implementation of the algorithm:

```
void findCycleDoubleCovers(List<Cycle> data, int start, Set<Edge>
    twoTimesUsedEdges) throws StopRecursionException {
        if (twoTimesUsedEdges.size() == graph.getEdges().size()) {
            CycleDoubleCover cycleDoubleCover = new CubicCycleDoubleCover(new
                HashSet<>(data));
            foundCDCs.add(cycleDoubleCover);
            if (any) { throw new StopRecursionException(); }
            return;
        }
        for (int i = start; i <= cycles.size() - 1; i++) {
            if (Collections.disjoint(twoTimesUsedEdges,
                cycles.get(i).getEdges())) {
                List<Cycle> newData = new ArrayList<>(data);
                newData.add(cycles.get(i));
                Set<Edge> newTwoTimesUsedEdges = new
                    HashSet<>(twoTimesUsedEdges);
                processTwoTimesUsedEdges(newData, newTwoTimesUsedEdges,
                    cycles.get(i));
                findCycleDoubleCovers(newData, i + 1, newTwoTimesUsedEdges);
            }
        }
    }
```

4.4 Verification of our conjecture

In addition to verifying our conjecture on the cubic graphs that we generated via Genreg [6], we used our software to help us to prove our conjecture on Issacs and Loupekine snarks (we were able to prove our theorem for generalized Blanuša snarks without the help of our software). These infinite graph classes are constructed by the cyclic connection of multipoles. Our approach was to find a semi-induced 6-CDC (an induced 7-CDC) of the smallest graph from the class and find a semi-induced 6-CDC (an induced 7-CDC) of the multipoles such that the cycles extend correctly and in the case of induced 7-CDC, such that there is no weak edge acquired. We found out that if we connect the corresponding dangling edges of two copies of $(3,3)$-poles Y, which we use to construct Issacs snarks, the obtained graph does not have an induced 7-CDC but four cyclically connected $(3,3)$-poles Y do have an induced 7-CDC, so we split the proof. We split the proof for generalized Blanuša and Loupekine snarks for the same reason (we divided the proof for generalized Blanuša snarks even though we had not known before that one such connected copy of the $(2,2)$-pole B does not have an induced 7-CDC).

4.4.1 Proving our conjecture on Issacs snarks

The induced 6-CDC that we used in the proof of our conjecture for Issacs snarks consisted of six circuits such that each circuit of the CDC passes through exactly one of the edges that we want to remove. Firstly, we were lucky to find such induced 6-CDC for J_{5+4k} for a non-negative integer k. As we know, if a graph has an induced 6-CDC, it implies that it has a semi-induced 6-CDC as well as an induced 7-CDC. Therefore, we can use an induced 6-CDC to prove both parts of our conjecture at once.

To find an induced 6-CDC for the J_{7+4k}, we excluded the induced circuits that do not pass through exactly one of the edges that we remove. We then divided the remaining induced circuits into three categories based on the removed edge that they pass through. To find an induced 6-CDC, instead of finding the right 6-combination of all circuits, we tried to find three 2-combinations of the circuits from the same category such that these three 2-combinations form an induced 6-CDC.

4.4.2 Proving our conjecture on Loupekine snarks

We tried the same approach, which we used on Issacs snarks, on Loupekine snarks, but we found out that small Loupekine snarks do not have an induced 6-CDC consisting of 6 induced circuits. We were not able to prove our conjecture on Loupekine snarks for L_{5+4k} for a non-negative integer k, because L_5 has 40 vertices, and our software cannot find a cycle double cover for such large graphs within a reasonable time.

Conclusion

In this thesis, we aimed to prove our own conjecture on some infinite graph families and discover new facts with regard to cycle double covers. We defined a circuit as a 2-regular connected graph and a cycle as a collection of edge-disjoint circuits. We also considered circuit double covers of graphs since each circuit is a cycle. The cycle double cover conjecture is equivalent to the assertion for cubic graphs [4]. Therefore, we only considered cubic graphs. We defined a weak and a strong edge in cycle double cover of cubic graphs. By Vizing's theorem [9], cubic graphs have chromatic index 3 or 4. We showed that the cycle double cover conjecture holds true for those with chromatic index 3. Such cubic graphs have a cycle double cover such that all edges are weak in it. We proved that an edge is weak in a circuit double cover if and only if the edge is a chord of a circuit of the circuit double cover. For cycle double covers, we showed that an edge is weak if and only if the edge is a chord of a circuit that comprises a cycle or the edge connects two circuits of the same cycle in the cycle double cover.

With regard to 2-edge cuts, we proved that the edges of a 2-edge cut are weak in any circuit double cover. We also proved that when we subdivide a 2-edge cut by an edge, the edge is weak in any cycle double cover. Concerning nontrivial 3-edge cuts, we showed that if two graphs have a semi-induced 6-CDC (an induced 7-CDC), then the graph obtained by a 3-cut-connection of these graphs also has a semi-induced 6-CDC (an induced 7-CDC). We showed that if a graph has a triangle, then each (semi-) induced CDC contains a cycle comprised of the triangle. We also pointed out that if a graph has a semi-induced 6-CDC (an induced 7-CDC), then the graph obtained by any inflation of a vertex has a semi-induced 6-CDC (an induced 7-CDC) as well. The same is true for the graph obtained by the E-inflation of an edge.

We proved our conjecture for Issacs snarks, generalized Blanuša snarks, and partially for Loupekine snarks. We also implemented software to find cycle double covers of cubic graphs and to help us to prove our conjecture. We accomplished the goals of our thesis. The future work might comprise proving our conjecture on other infinite graphs families and implementing a faster software to verify our conjecture.

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