Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Analysis of colouring properties of proper (2, 3)-poles

Bachelor's Thesis

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Faculty of Mathematics, Physics and Informatics

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Analýza farebných vlastností vlastných (2, 3)-pólov

Anotácia: Snarky, teda 3-regulárne grafy nemajúce hranové 3-farbenie, sú intenzívne skúmanou triedou grafov, nakoľko obsahujú potenciálne najmenšie protipríklady k mnohým dlhoročne otvoreným problémom. Ak snark rozdelíme hranovým 5-rezom na dva zafarbiteľné komponenty, možno ich na základe ich farebných vlastostí klasifikovať ako superpentagony, negátory alebo vlastné (2, 3)-póly. Zatiaľ čo farebné vlastnosti prvých dvoch typov sú dobre preskúmané, o vlastných (2, 3)-póloch je toho známe najmenej. Majú najviac možností, ako môže vyzerať množina ich prípustných farebných typov a nie sú známe nutné či postačujúce podmienky, kedy majú farbenie všetkých dovolených typov. Cieľom tejto práce je vniesť do tejto problematiky viac svetla. Práca predpokladá využitie výpočtovej techniky pre analýzu malých prípadov a následné zovšeobecnenie do teoretických výsledkov.

Cieľ: 1. Za pomoci počítačových nástrojov analyzovať farebné vlastnosti vlastných (2, 3)-pólov, ktoré v znikajú z malých snarkov.
2. Na základe výsledkov v malých prípadoch sformulovať a dokázať tvrdenia (nutné podmienky a postačujúce podmienky) o farebných vlastnostiach vlastných (2, 3)-pólov, najmä tvrdenia o tom, kedy je vlastný (2, 3)-pól perfektný.

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Annotation: Snarks, that is 3-regular graphs admitting no 3-edge-colouring, are an intensively researched class of graphs, because they contain potential smallest counterexamples to many long-standing open problems. If we disconnect a snark by a 5-edge cut into two colourable components, it is possible to classify them based on their colouring properties as superpentagons, negators, or proper (2,3)-poles. While the colouring properties of the first two types are well studied, those of proper (2,3)-poles are the least known. They have the most possibilities of how the set of their permissible colouring types, and there are no known necessary or sufficient conditions when they have the colouring of all permitted types. The goal of this thesis is to shed more light on this issue. The thesis assumes the use of computing technology for the analysis of small cases and subsequent generalization into theoretical results.

Aim: 1. With the help of computer tools, analyse the colouring properties of proper (2,3)-poles that originate from small snarks.
2. Based on the results in small cases find out and formulate propositions (necessary and sufficient conditions) about various properties of proper (2,3)-poles, especially propositions about when a proper (2,3)-pole is perfect.

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Abstrakt

Snark je bezmostový neorientovaný kubický graf, ktorého hrany sa nedajú zafarbiť troma farbami tak, že každá dvojica susedných hrán má rozdielne farby. Niekedy nepožadujeme, aby každá hrana v grafe bola incidentná s práve dvoma vrcholmi, ale povoľujeme aj s jedným alebo žiadnym. Takéto štruktúry sa nazývajú multipóly a vieme ich spájať dokopy, vytvárajúc väčšie multipóly, ako aj grafy. V našej práci skúmame farebné vlastnosti vlastných (2,3)-pólov, ktoré sú špecifickým typom multipólov s piatimi visiacimi hranami, čiže hranami incidentnými s práve jedným vrcholom. Vznikajú zo snarkov odstránením vrcholu a prerezaním hrany nie incidentnej s odstráneným vrcholom. Naša analýza zahŕňa preskúmanie všetkých vlastných (2,3)-pólov, ktoré vznikajú zo snarkov s obvodom aspoň päť a s maximálne 28 vrcholmi. Dokopy ide o 3 247 snarkov a 3 476 400 vlastných (2,3)-pólov. V našom výskume predstavujeme rôzne štruktúry, ktoré možno využiť na rozšírenie farebnosti vlastných (2,3)-pólov. V hlavnej časti našej práce poskytujeme vety týkajúce sa farebných vlastností vlastných (2,3)-pólov, konkrétne nutné a postačujúce podmienky týkajúce sa týchto farebných vlastností. Okrem toho prezentujeme dáta a pozorovania z analýzy, spolu s niektorými problémami pre ďalší výskum.

Kľúčové slová: snark, multipól, hranové farbenie, Taitovo farbenie, množina farbení

Abstract

Snark is a bridgeless undirected cubic graph whose edges cannot be coloured with three colours so that no two adjacent edges have the same colour. Sometimes, we do not require each edge to be incident with precisely two vertices but instead, allow even one or none. Structures allowing this are called multipoles and can be joined together through junctions, forming larger multipoles and even graphs. In our work, we explore the colouring properties of proper (2,3)-poles, a specific type of multipole with five dangling edges, that is edges that are incident with only one vertex. They result from snarks by removing a vertex and severing an edge, not incident with the removed vertex. To conduct our analysis, we explore all proper (2,3)-poles resulting from nontrivial snarks with girth at least five and with at most 28 vertices. This encompasses a total of 3,247 snarks and 3,476,400 proper (2,3)-poles. In our research, we provide various structures that can be utilized to expand the colourability of proper (2,3)-poles. In the core of our work, we provide theorems regarding the colouring properties of proper (2,3)-poles, specifically necessary and sufficient conditions for these properties. Additionally, we present the data and observations from the analysis, along with some problems for further research.

Keywords: snark, multipole, edge-colouring, Tait colouring, colouring set

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Introduction

The study of *snarks*, that is bridgeless cubic graphs that are not 3-edge-colourable, is important since they are smallest possible counterexamples for several open problems. The Four Colour Theorem may be the most famous problem, stating that the regions of any simple planar map can be coloured with only four colours in such a way that every two adjacent regions sharing a border are coloured with different colours. There were many attempts to prove this theorem until 1976 when K. Appel and W. Haken proved it with the help of a computer [1]. Since the proof is computer-assisted and thus hard for a human to check by hand, some mathematicians still need to accept this proof. That is where P. G. Tait and his attempt at the proof from 1880 comes in. He proved that this theorem is equivalent to the statement that every bridgeless cubic planar graph is 3-edge-colourable. Another example of such a problem may be the Cycle Double Cover Conjecture, stating that each bridgeless graph has a family of cycles, such that each edge appears in exactly two of the cycles. It is proven that the minimum counterexample must be a snark [2]. The term snark was first used in 1976 by M. Gardner [3].

Determining whether a given graph is a snark is an NP-complete problem [4]. However, when creating a cubic graph from some smaller building blocks, of which we know their colouring properties, we also know the colouring properties of the result. This may also include if it is a snark or not. These building blocks are called multipoles and can be described as an extended type of a graph that allows each edge to be incident with two or fewer vertices. Each edge has two edge ends, which may or need not be incident with some vertex. If some edge is incident with only one vertex, we say it is *dangling*, and if with none, then it is *isolated*. The edge ends not incident with any vertex are called *semiedges*.

Usually, multipoles are constructed from cubic graphs by removing some vertices or severing some edges. In our research, removing a vertex from a graph involves keeping the edges incident with the vertex but making them dangling. This means that the edge ends that were previously incident with the removed vertex are no longer incident with any vertex, thus resulting in a multipole. Similarly, severing an edge involves replacing it with two new dangling edges from its end vertices. Using these dangling edges, we can connect multiple multipoles to create bigger building blocks or even graphs. This means that the analysis of the colouring properties of these multipoles can aid in the study and construction of snarks.

By talking about proper (2,3)-poles, we mean a specific type of a multipole with five dangling edges (usually named a 5-pole), for which we can divide the semiedges into two so-called connectors, one with two semiedges and one with three. Also, they fulfil the condition of being proper (Section 1.3). In general, by k-poles, we denote multipoles that have exactly k semiedges. Proper (2,3)-poles can be created by removing a vertex from a snark and severing one edge.

Exploring multipoles effectively involves starting with the simplest ones and gradually moving towards more complex ones. That's why we have chosen to investigate the k-poles starting with the smallest value of k. The reason we are exploring 5-poles is that for each $k < 5$, the colourability of k-poles has already been investigated. Specifically, the colouring properties of 1-poles, 2-poles, and 3-poles are trivial, and the colouring properties of 4-poles are limited and have already been sufficiently explored.

It can be proven that if we get two 3-edge-colourable 5-poles by severing five edges in a given snark, they can be only one of three types: negators, superpentagons and proper $(2,3)$-poles. The colouring properties of the first two are known, only the last type of 5-poles still needs to be explored, and thus we want to shed some light on them as well.

Our work analyses the colouring properties of proper (2,3)-poles originating from small snarks. Based on the analysis results, we then formulate propositions on sufficient and necessary conditions on the colouring properties of proper (2,3)-poles.

This thesis is structured as follows: In Chapter 1, we define and explain the terms necessary for understanding this topic, like the mentioned snarks, multipoles and others. In Chapter 2, we explain what the proper (2,3)-poles are. We define the mentioned colouring properties and divide proper (2,3)-poles into classes based on their colouring properties. In Chapter 3, we present the results of our work. To begin, we explain the methods of analysis we used, including a simplified explanation of the algorithm and the format of the outputs. We also provide some multipoles that can modify the colouring properties of proper (2,3)-poles. The following section comprises the core of our work, presenting the theorems regarding the colouring properties of proper (2,3)-poles. Specifically, we explore the necessary and sufficient conditions for these properties. Lastly, we provide data and observations from our analysis, including problems and questions that arose while exploring proper (2,3)-poles. We answer some of them but also include unanswered questions and problems that can be explored in further research.

Chapter 1

Multipoles and Snarks

1.1 Multipoles

All graphs considered in this work are undirected, and while we permit multiple edges, loops are not allowed. Definitions not provided in our work can be found in the book "Graph Theory" by R. Diestel [5].

The distance between two vertices x and y in a graph G, denoted by $d_G(x, y)$, is defined as the length of the shortest path between x and y in G. If no such path exists, we set $d_G(x, y) = \infty$. For a vertex x and an edge ab in G, the distance is defined as the smallest value between $d_G(x, a)$ and $d_G(x, b)$.

In our work, we allow a specific modification of graphs, where the ends of its edges may not be incident with a vertex, resulting in a graph with dangling edges. Such structures are called multipoles.

Definition 1. A *multipole* is a pair $M = (V, E)$ of distinct finite sets of vertices V and edges E, where every edge $e \in E$ has two edge ends, which may or need not be incident with a vertex.

According to the incidence of edge ends, we define four types of edges:

1. A *link* is an edge whose ends are incident with two distinct vertices.
2. A *loop* is an edge whose both ends are incident with the same vertex.
3. A *dangling edge* is an edge whose one end is incident with a vertex, and the other is not.
4. An *isolated edge* is an edge whose both ends are not incident with any vertex.

Loops are not allowed in our work, although, for the sake of a complete definition, they are included in the types of edges. We define some properties of multipoles based on their edge ends, but at first, it is essential to define the edge ends, not incident with a vertex.

Definition 2. A *semiedge* is an edge end not incident with any vertex. For a given multipole M, we define $S(M)$ to be the tuple containing all the semiedges in that multipole.

A multipole M with $S(M) = (a_1, \cdots, a_n)$ can also be denoted as $M(a_1, \cdots, a_n)$. The *order* of a multipole M, denoted by $|M|$, is the number of its vertices. The *degree* of a vertex v of a multipole, denoted by $\deg(v)$, is the number of edge ends incident with v. In our work, we will consider *cubic* multipoles, i.e. multipoles where every vertex has degree 3.

A multipole with k semiedges is usually called a *k-pole*. Based on this definition, it is possible to define a graph as a multipole without semiedges, or more precisely, a *0-pole*.

One of the features of multipoles is connecting them to form bigger multipoles or even graphs. They can be seen as small building blocks for constructing larger graphs or multipoles. For this case, dividing $S(M)$ into pairwise disjoint tuples called connectors is convenient.

Definition 3. Let M be a multipole. Parts of the partition of $S(M)$ into pairwise disjoint tuples $S_1, \cdots, S_n$ are called *connectors*.

A multipole M with n connectors $S_1, \cdots, S_n$, where S_i has c_i semiedges for each i from 1 to n, is denoted by $M(S_1, \cdots, S_n)$ and is also called a $(c_1, \cdots, c_n)$-pole.

Now, let e and f be edges (not necessarily distinct) and e', f' two of their semiedges respectively, such that $e' \neq f'$. If $e \neq f$, the result of the *junction* of e' and f' is a new edge, having the other edge ends of e and f and a deletion of e and f. If $e = f$, the result of the *junction* of e' and f' is just the deletion of the edge.

Let $S = (e_1, \cdots, e_n)$ and $T = (f_1, \cdots, f_n)$ be two connectors, both with n semiedges. The junction of two connectors S and T consists of n individual junctions of semiedges e_i and f_i for i from 1 to n.

The junction of two $(c_1, \cdots, c_n)$-poles $M(S_1, \cdots, S_n)$ and $N(T_1, \cdots, T_n)$ consists of n individual junctions of connectors S_i and T_i, for i from 1 to n.

Consider two multipoles $M(a_1, \cdots, a_n)$ and $N(b_1, \cdots, b_m)$. Their *partial junction* is a junction of some semiedges $(a_{i_1}, \cdots, a_{i_k})$ and $(b_{j_1}, \cdots, b_{j_k})$, where $k \leq n$ and $k \leq m$. In contrast to a normal junction, which results in a graph, the partial junction can still result in a multipole.

Let G be a graph, ab its edge, and v its vertex. By *severing* the edge ab in G, we mean removing ab and adding a dangling edge to the vertices a and b. Similarly, *removing* the vertex v involves the removal of v along with all of its incident edges, followed by adding a dangling edge to all of the formerly neighbouring vertices of v. If we obtain a multipole by removing some vertices and severing some edges in a graph, there is a default way to divide the resulting semiedges into connectors. When we remove a vertex, all semiedges formerly incident with the vertex are in a new connector. Similarly, when we sever an edge, the two new semiedges are in a new connector.

To properly denote the multipoles resulting from a graph by removing some vertices and severing some edges, we will denote such multipoles as $R(G;V;E)$, where G is the former graph, V is the set of removed vertices, and E is the set of severed edges. For example, a multipole resulting from a snark G by removing vertex v and severing edge ab is denoted by $R(G;\{v\};\{ab\})$ and consists of two connectors, one with two semiedges and one with three. In the case where a set contains only one element, we can represent it without brackets, resulting in this case in the notation $R(G;v;ab)$.

1.2 Snarks

As mentioned in Introduction, the study of the problem of 3-edge-colourability of cubic graphs is important since they are minimal possible counterexamples to several open problems.

An *edge-colouring* of a graph is a mapping ϕ from the set E of edges of the graph to some non-empty set of colours C such that every two adjacent edges have assigned different elements from C. Subsequently, a graph is *n-edge-colourable* if and only if there exists an edge-colouring of the graph, where the set of colours has precisely n elements.

Graph G is called *k-connected* if $|G| > k$ and $R(G;X;\emptyset)$ is connected for every set $X \subseteq V,\ |X| < k$ [5].

Definition 4. *Snark* is a 2-connected cubic graph which is not 3-edge-colourable.

The term snark comes from the article by M. Gardner from 1976 [3]. However, the definition used there slightly differs from the one we use in our work (Definition 4).

From now on, if we say that a graph or a multipole is colourable, we mean that there exists a 3-edge-colouring for that given construction.

Let G be a cubic graph and S an edge-cut of size n. By severing the edges of S, we obtain two n-poles such that G is a junction of them. If both contain a cycle, S is said to be an *n-edge-c-cut*. Generally, these edge cuts are called *c-cuts*.

A cubic graph G is called *cyclically n-edge-connected* if there is no c-cut with less than n edges.

Definition 5. Let G be a cubic graph with at least one c-cut. *Cyclic edge connectivity* of G is the smallest number of edges of a c-cut of G and is denoted by $z(G)$.

Another attribute of graphs that shall be defined is the *girth* of a graph G, which is the minimum length of a cycle in G. If G does not contain a cycle, we set the girth to ∞.

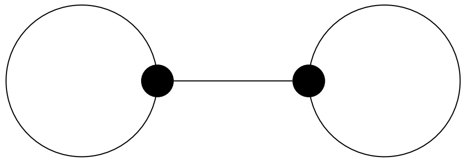


Figure 1.1: Dumbbell graph

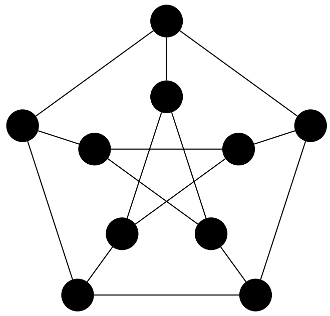


Figure 1.2: Petersen graph

Many authors include in their definition of snarks additional criteria of "non-triviality", for example, girth at least five or being cyclically-edge-4-connected [6, 7]. On the other hand, some authors allow snarks to contain bridges [8], making the dumbbell graph on two vertices the smallest snark (Figure 1.1). We discuss more about the triviality in Section 1.4.

The Petersen graph is the smallest snark satisfying every definition (Figure 1.2). Other notable snarks are the Blanuša snarks, or the infinite family of flower snarks discovered by R. Isaacs [9]. The Isaacs snarks are denoted by J_k, where k is an odd integer $k \geq 3$.

The term "snark hunting", used by M. Gardner, is more than justified since the proportion of snarks in all cubic graphs is really small because almost every cubic graph is Hamiltonian [10], and hence colourable, as well as determining whether a cubic graph is colourable or not is an NP-complete problem [4]. Because of that, sometimes we want to create snarks by the junction of some multipoles, and by knowing their colouring properties, we immediately know if the result is a snark or not.

1.3 Multipole Colouring

Definition 6. Let G be a graph and let Γ be a group. A nowhere zero Γ-flow of G is a pair (D, f), where D is an orientation assignment for each edge and f is a function $f\colon E(G) \to \Gamma$ such that the following conditions hold:

1. For every edge e in the set $E(G)$, the function f assigns a non-zero element, i.e., $f(e) \neq 0$.

2. For each vertex $v \in V(G)$, the value of $\sum_{e \in v^+} f(e) - \sum_{e \in v^-} f(e)$, where v^+ and v^- represent the sets of the edges entering and exiting vertex v, respectively, with orientations determined by D, is equal to zero.

Since we are only exploring 3-edge-colourings, it may be convenient to define some set of colours to be used in each colouring in our work. The set we shall use is that of non-zero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we will denote as $\mathbb{K}$. These colourings are called *Tait colourings*. These are widely used in many articles about snarks and 3-edge-colourability in general because the problem of colourability is equivalent to the problem of finding a nowhere zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow. Since each non-zero element from this group is self-inverse, we need not assign an orientation to the edges. The only way to achieve a zero-sum in a vertex using this set is to add three distinct values. If we were to use two identical values, we would need a third value to be zero, but since we only use non-zero elements, this is impossible. Also, the sum of the non-zero elements indeed equals zero. From now on, by default, each mentioned colouring will be a Tait colouring if not defined otherwise. Sometimes for better readability, we will use colours $1, 2, 3$ instead of $(0,1), (1,0), (1,1)$, respectively.

When discussing multipoles, the definition of edge-colouring is the same. The colour of an edge end is the colour of its respective edge. Now, we can introduce a new term: *colouring set*.

Definition 7. Let $M(e_1, \cdots, e_k)$ be a k-pole. The colouring set of M is defined as follows:

$$\mathrm{Col}(M) = \{(\phi(e_1), \cdots, \phi(e_k)) \mid \phi \text{ is a Tait colouring of } M\},$$

so a set of k-tuples representing all possible colourings of the semiedges of M.

As mentioned, Tait colourings are equivalent to nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows. For a better exploration of the colours in separate connectors, we shall define what is a *flow through a connector*.

Definition 8. Let M be a multipole, ϕ a Tait colouring of M and $S = (e_1, \cdots, e_n)$ a connector of M. Then a *flow through* S is defined as $\phi_*(S) = \sum_{i=1}^{n} \phi(e_i)$.

In general, for any tuple of semiedges $S = (e_1, \cdots, e_n)$ and colouring ϕ, we use the notation $\phi(S)$ to represent the tuple $(\phi(e_1), \cdots, \phi(e_n))$. By leveraging our understanding of Tait colourings, we can define the term *proper*, which is closely linked to colourability.

Definition 9. A connector S of a multipole M is called *proper* if $\phi_*(S) = 0$ for each Tait colouring ϕ of M. A multipole is called *proper* if each of its connectors is proper.

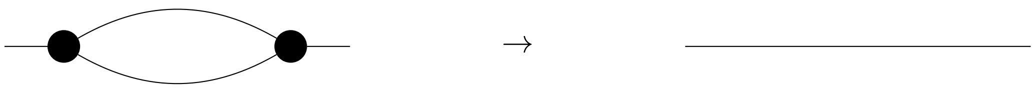


Figure 1.3: Replacement of a digon in a snark

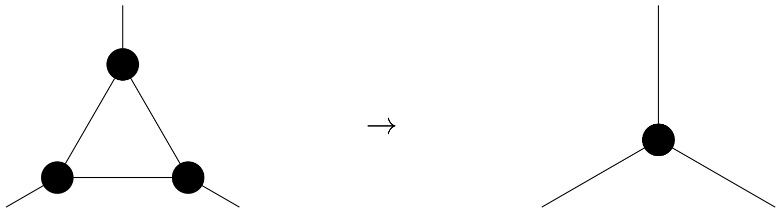


Figure 1.4: Replacement of a triangle in a snark

The fact that we can regard a colouring of a multipole as a flow has a valuable consequence that will be indispensable in our work. It is commonly known as the Parity Lemma, introduced and proved by B. Descartes in 1948.

Lemma 1 (Parity Lemma [11]). *Let M be a k-pole, and let k_1, k_2 and k_3 be the numbers of semiedges of colour* $(0,1)$*,* $(1,0)$ *and* $(1,1)$*, respectively. Then* $k_1 \equiv k_2 \equiv k_3 \mod 2$.

By applying the Parity Lemma, we can conclude that any cubic graph with a bridge is not colourable. Another corollary of this lemma is that the minimum number of vertices that must be removed from a snark to obtain a colourable multipole is two [12]. The same applies to severing edges. The smallest number of edges to be severed in a snark to obtain a colourable multipole is two. If only one edge is severed, the resulting multipole contains two semiedges, both of which must have the same colour for it to be colourable because of the Parity Lemma. That would mean the former graph resulting from the junction of these two semiedges is not a snark since the colouring of the multipole could be extended to the colouring of the graph.

1.4 Triviality of Snarks

As mentioned earlier, some authors use the term "nontrivial" when defining snarks. One of the attributes linked to the triviality of snarks is the containment of a digon, a graph with two vertices connected by two edges. Consider a graph G with a digon. Let G' be a graph obtained by removing the digon, as shown in Figure 1.3. The graph G is colourable if and only if G' is colourable.

If a cubic graph G contains a triangle, it can be replaced with a single vertex resulting in a smaller graph G' as visualised in Figure 1.4. As with digons, G is colourable if and only if G' is colourable.

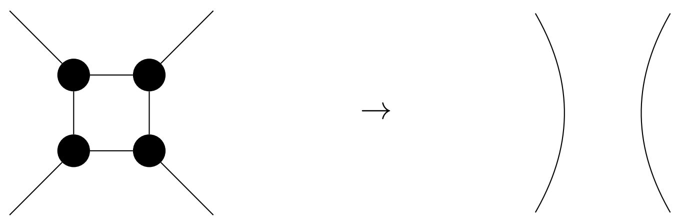


Figure 1.5: Replacement of a quadrilateral in a snark

A less evident attribute could be the containment of a quadrilateral in a graph G. As shown in Figure 1.5, it can be replaced with two edges, resulting in a graph G'. If G' is colourable, G is also colourable. It must be noted, though, that the converse implication does not hold. In general, based on these three attributes, we can mark snarks containing digons, triangles or quadrilaterals as trivial.

As mentioned before, a cubic graph with a bridge is uncolourable. If a graph has cyclic edge connectivity 1, it is clear that it has a bridge, thus, is uncolourable. Because of that, we consider the cubic graphs with cyclic edge connectivity 1 as trivial snarks. Now let us consider a snark G with cyclic edge connectivity 2. That means it has a 2-edge-cut, decomposing it into two 2-poles, M and N. We show that at least one of them is uncolourable. Suppose that both are colourable. Because of the Parity Lemma, both semiedges in M and N have the same colour, so after the junction of $S(M)$ and $S(N)$, we obtain G, which would be colourable and that is impossible. Therefore at least one of them is uncolourable, which means we can obtain a new smaller snark by the junction of the two semiedges in the uncolourable 2-pole.

Assume that a snark G has cyclic edge connectivity 3. As before, at least one of the components M, N obtained by severing the edges in the 3-edge-cut of G must be uncolourable, otherwise, all three semiedges of M and N would have three different colours because of the Parity Lemma, and thus the colouring of G could be obtained. This means that we can construct a smaller snark by joining the semiedges of the uncolourable component in a new vertex.

In our work, a snark is considered trivial if a new smaller snark can be obtained using one of the methods mentioned above in this chapter. Now we can adequately define when a snark is nontrivial.

Definition 10. A snark is called *nontrivial* if it has girth at least five and is cyclically 4-edge-connected. Otherwise, it is called *trivial.*

Chapter 2

Proper (2,3)-poles

Using the definitions from the previous chapter, it may be clear what the name proper (2,3)-pole means. It is a proper multipole consisting of two connectors, one containing two semiedges and the second three. We can easily construct a snark from it by the junction of the semiedges in the connector of size two and joining the semiedges from the connector of size three to a new vertex, creating an edge and a vertex, respectively. Since they are proper, the result is a snark because there is no Tait colouring, where the edges joined in the new vertex would have all different colours. Similarly, proper (2,3)-poles do not allow Tait colourings, in which the semiedges in the connector of size 2 have the same colour.

Thus if we take a cubic graph G, remove its vertex v and sever its edge ab, which is not incident with v, and the result is a proper (2,3)-pole, then G must be a snark. Now let us look at it the other way and prove that for each snark, the result after removing a vertex and severing an edge will always be a proper (2,3)-pole.

Lemma 2. *Let G be a snark. Let $T(A, B)$ be a multipole $R(G; v; ab)$ where the edge ab is not incident with v, such that A contains the two semiedges resulting from severing ab and connector B contains the three semiedges resulting from removing v. Then $T(A, B)$ is a proper (2,3)-pole.*

Proof. We prove that $T(A, B)$ is a proper (2,3)-pole. Suppose the contrary, so there is a colouring ϕ, for which $\phi_*(A) = 0$ or $\phi_*(B) = 0$. Then by the Parity Lemma, the flow through the second connector is also zero, so $\phi_*(A) = 0, \phi_*(B) = 0$. This allows extending the multipole by joining the semiedges in the connector B and performing the junction of semiedges in the connector A, resulting in the original graph, which is colourable. This means that G is not a snark, which is a contradiction. □

By default, when we get a proper (2,3)-pole in this way, we denote it by $T(A, B)$, where the connector A contains the two semiedges resulting from severing the edge and similarly, connector B contains the three semiedges resulting from removing the vertex,

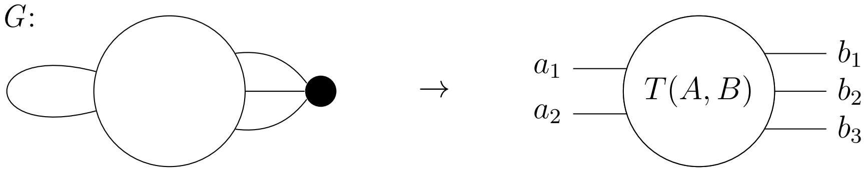


Figure 2.1: Creation of a proper $(2,3)$-pole from a snark G

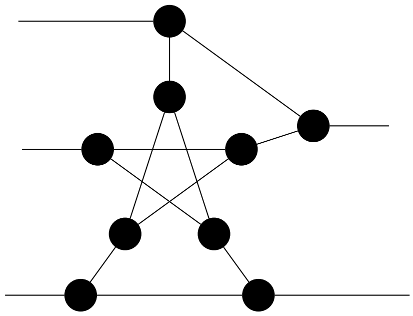


Figure 2.2: Creation of a proper $(2,3)$-pole from the Petersen graph

such that $A = (a_1, a_2)$ and $B = (b_1, b_2, b_3)$. An example can be seen in Figure 2.1. A more specific example can be seen in Figure 2.2, where the proper $(2,3)$-pole results from the Petersen graph.

To explore multipoles effectively, it is best to start with the simplest ones and gradually move towards more complex ones. That's why we begin by examining the k-poles starting from the smallest k. Specifically, the smallest k for which it is interesting to explore the colourability of k-poles arising from snarks with k-edge-cuts is 4. The 1-poles are trivially uncolourable, while the colourability of 2-poles and 3-poles is limited due to the Parity Lemma. For a 2-pole to be colourable, both of its semiedges must have the same colour. Similarly, for a 3-pole, all three of its semiedges must have pairwise different colours. Also, the colouring properties of 4-poles are already widely explored since their colouring properties are limited as well [13]. That is why we have moved to explore 5-poles. The motivation to explore specifically the colouring properties of proper (2,3)-poles comes from the fact that proper (2,3)-poles are one of three types of interesting 5-poles resulting from snarks, based on their colouring properties. We can introduce a theorem from an article by P. J. Cameron, A. G. Chetwynd and J. J. Watkins, proving that this is true.

Theorem 1 ([14]). *Let G be a snark with a 5-edge-c-cut whose removal results in components G_1 and G_2, and G is not the Petersen graph. Then either*

1. *one of G_1, G_2 is not 3-edge-colourable, or*
2. *both can be extended to snarks by adding at most five vertices such that at least one of those extended graphs is smaller than G.*

Using this theorem and the Parity lemma, it can be proven that if we get two colourable 5-poles by severing five edges in a given snark, they can be only one of three types: negators, superpentagons and proper $(2,3)$-poles. For the first two, their colouring properties are already explored, which is why we chose to explore the last type.

Before introducing some of the mentioned theorems, we shall define what it means when some pair of vertices or edges is removable.

Definition 11. Let G be a snark. A pair of its distinct vertices $\{u, v\}$ is called *removable* if $R(G; \{z, v\}; \emptyset)$ is not colourable; otherwise, it is called *unremovable.* Similarly, for edges, a pair of distinct edges $\{ab, cd\}$ is called *removable* if $R(G; \emptyset; \{ab, cd\})$ is colourable. Otherwise, it is called *unremovable.*

If we sever two adjacent edges, it is equivalent to the removal of a single vertex with regard to colourability. Therefore these pairs of edges are trivially removable because at least two vertices are needed to be removed from a snark to obtain a colourable multipole.

Negator $Neg(G; u, v)$ is a 5-pole resulting from the removal of a path uwv of length 2 from a snark G (w is a common neighbour of u and v). By default, negators have three connectors: $I = (e_1, e_2)$ and $O = (e_3, e_4)$ the connectors consisting of the semiedges formerly incident with u and v, respectively, and $R = (e_5)$ the connector containing the remaining semiedge. The colouring set of each negator is a subset of

$$C = \{(x, x, a, b, a + b), (a, b, x, x, a + b) \in \mathbb{K}^5 \mid a \neq b\} \text{ [12].}$$

A negator whose colouring set is identical to C is called *perfect*; otherwise, it is called *imperfect.* For an *imperfect* negator, it is possible that one of its connectors of size two is improper, which means that the other connector of size 2 is proper. If such a negator additionally admits all such colourings, it is called *semiperfect* [12]. The following is the theorem about the negators and when they are perfect or semiperfect.

Theorem 2 ([8])**.** *Let $N = Neg(G; u, v)$ be a negator and let w be a common neighbour of u and v. If N is colourable, then it is either perfect or semiperfect. Moreover, the following hold.*

1. *N is perfect if and only if each of the pairs $\{u, w\}$ and $\{v, w\}$ is unremovable.*
2. *N is semiperfect if and only if at least one of the pairs $\{u, w\}$ and $\{v, w\}$ is removable.*

Let $C_5 = C_5(e_0, \cdots, e_4)$ denote the 5-pole consisting of a 5-cycle having vertices $v0, \cdots, v_4$, arranged cyclically, with five semidges $e_0, \cdots, e_4$ attached to them correspondingly. Superpentagon is any 5-pole M with $\mathrm{Col}(M) \subseteq \mathrm{Col}(C_5)$ [12].

For a 5-pole M with $\mathrm{Col}(M) \subseteq \mathrm{Col}(C_5)$ only two possibilities can occur: either $\mathrm{Col}(M) = \emptyset$ or $\mathrm{Col}(M) = \mathrm{Col}(C_5)$ [12]. In the latter case, we call M a *perfect* superpentagon. An example of a perfect superpentagon has 15 vertices and can be obtained from the Isaacs flower snark J_5 by removing the unique 5-cycle of J_5.

2.1 Colouring set

By applying the knowledge about the multipoles being proper and the Parity lemma, it is clear that the colouring set of each proper (2,3)-pole is a subset of

$$C = \{(a_1, a_2, b_1, b_2, b_3) \in \mathbb{K}^5 \mid a_1 + a_2 = b_1 + b_2 + b_3 \neq 0\}.$$

We will refer to this set only as C from now on. Note that in this definition, the terms a_1 to b_3 denote the colours of the respective semiedges. The non-equality to zero is evident because they are proper, and the equality of flow through both connectors follows from the Parity lemma.

2.2 Perfect (2,3)-poles

Interesting types of proper (2,3)-poles are those whose colouring set is the same as C defined above since they admit all possible colourings. We can define the term *perfect* used to label these.

Definition 12. A proper (2,3)-pole is called *perfect* if its colouring set coincides with the colouring set C.

Another corollary of the Parity Lemma may be that for each 5-pole to be colourable, it needs to have three semiedges of one colour and the other two with a different colour each. This way, we can define so-called solitary and sociable semiedges.

Definition 13. Let M be a 5-pole and ϕ its colouring. The three semiedges coloured by the same colour are called *sociable* in ϕ, and the other two coloured by a colour different from the others are called *solitary* in ϕ.

There may seem to be many possible colourings of proper (2,3)-poles, as, in theory, each subset of C represents a different colouring set. However, we will show that not each of these subsets can be attained.

Let M be a multipole, ϕ_1 and ϕ_2 colourings of M such that ϕ_1 uses $C = \{c_1, c_2, c_3\}$ as the set of colours and ϕ_2 uses $D = \{d_1, d_2, d_3\}$. We say that ϕ_1 and ϕ_2 are *isomorphic* if there exists a bijection $f\colon C \to D$ and for each edge e from $E(M)$ $\phi_2(e) = f(\phi_1(e))$.

Let T be any proper (2,3)-pole. First, we can see that because of colouring isomorphism, if one colouring can be attained, then the colouring set of T must contain this colouring along with all possible isomorphic colourings to the first one. Also, we can get a new colouring using the interchange of colours on a chain.

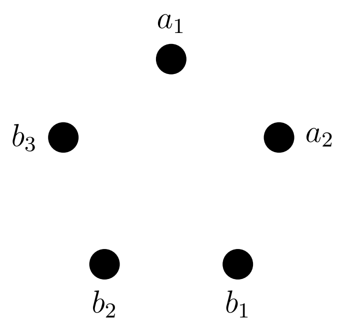


Figure 2.3: Vertices of a colouring graph of proper (2,3)-poles

Lemma 3 ([6]). *Let T be a 5-pole and ϕ a colouring of T. Let p and q be its two solitary semiedges in ϕ. Then there exists a sociable semiedge p' and another colouring of T, in which p' and q are solitary.*

The idea behind the proof is that if p has colour 1 and q colour 2, we take into account a subgraph of T formed by the edges with colours 1 and 3. In this partial graph, there is a path, which is called a (Kempe) chain, in which the extremities are p and a sociable semiedge p'. By the interchange of colours 1 and 3 along this chain, we obtain a new colouring of T in which p' and q are solitary.

Because of the isomorphism of colourings, it does not matter which colours we use in the colouring, but rather which semiedges are solitary. Using this, we can visualize the colouring set of any 5-pole in the following way.

For a 5-pole $T(e_1, e_2, e_3, e_4, e_5)$ we denote by R_T a graph with vertex set $V = \{e_1, e_2, e_3, e_4, e_5\}$, in which for each $e_i, e_j \in V$, the graph contains an edge e_ie_j if and only if there exists a colouring ϕ of T such that the semiedges e_i and e_j are solitary in ϕ. R_T is called a *colouring graph of* T. By Lemma 3, R_T has no pendant vertex [6].

Since the semiedges in proper (2,3)-poles are $(a_1, a_2, b_1, b_2, b_3)$, we can use them as the set of vertices in the colouring graphs of proper (2,3)-poles. These vertices can be seen in Figure 2.3.

We will introduce a new notation to help explain how the colouring graph of a 5-pole looks. Let T be a 5-pole. We say that T *allows solitary cycle* $e_1e_2 \cdots e_n$ if for each i from 1 to $n-1$, T allows a colouring, where e_i and e_{i+1} are solitary, including the colouring where e_n and e_1 are solitary.

2.3 Colouring classes

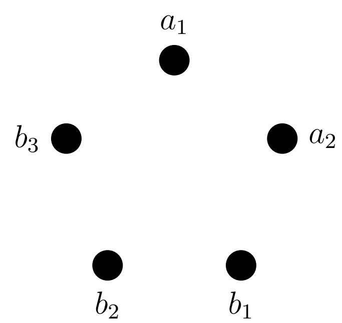
Using the knowledge from Section 2.2, we can visualize all possible colouring sets of proper (2,3)-poles on their colouring graphs. Since there is no rule on which semiedges are labelled b_1, b_2, b_3 after removing a vertex from a snark, we will divide the colouring sets into classes, in which the colouring sets represent isomorphic colourings up to the choice of labelling b_1, b_2, b_3.

However, not each graph on these vertices represents a possible colouring set. First, the graph must have no pendant vertex, as was proven above. The second restriction is that no edge is between two vertices from $\{b_1, b_2, b_3\}$. Suppose there was an edge between b_1 and b_2 in any proper (2,3)-pole T. This would mean that T allows a colouring such that b_1, b_2 and b_3 have pairwise different colours and a_1, a_2 have the same colour. Thus T could be extended to the former graph, which would be colourable and therefore not a snark, which contradicts the assumption.

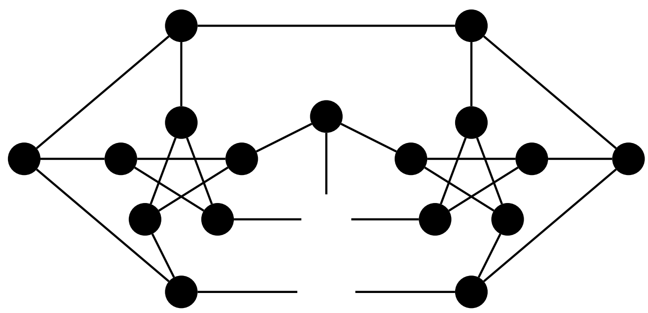
This means there are only 12 different colouring sets of proper (2,3)-poles, which can be divided into classes in the following way. We denote the classes by a number representing how many vertices from $\{b_1, b_2, b_3\}$ are connected to $\{a_1, a_2\}$ with an edge in the colouring graph, followed by A if the colouring allows an edge between a_1 and a_2, or B otherwise.

2.3.1 Uncolourable

Even after removing a vertex and severing an edge, the resulting multipole can still be uncolourable. Its colouring graph has thus no edges and can be seen in Figure 2.4a. In our notation, this would be named 0B. An example of an uncolourable proper (2,3)-pole constructed from the second Blanuša snark can be seen in Figure 2.4b.



(a) Colouring graph of uncolourable 5-poles



(b) Example of an uncolourable proper (2,3)-pole

Figure 2.4: Uncolourable proper (2,3)-poles

2.3.2 Class 1A

In this class, colouring sets allow only one semiedge from $\{b_1, b_2, b_3\}$ to be solitary along with a_1, a_2, including an option where a_1 and a_2 are solitary. Note that we cannot obtain a class 1B because the vertices a_1 and a_2 in its colouring graph would be pendant. All three possibilities are visualized in Figure 2.5. An example of a proper (2,3)-pole from class 1A can be seen in Figure 2.6, resulting from the Petersen graph, by severing an edge and removing a vertex with a distance of 1 from the edge.

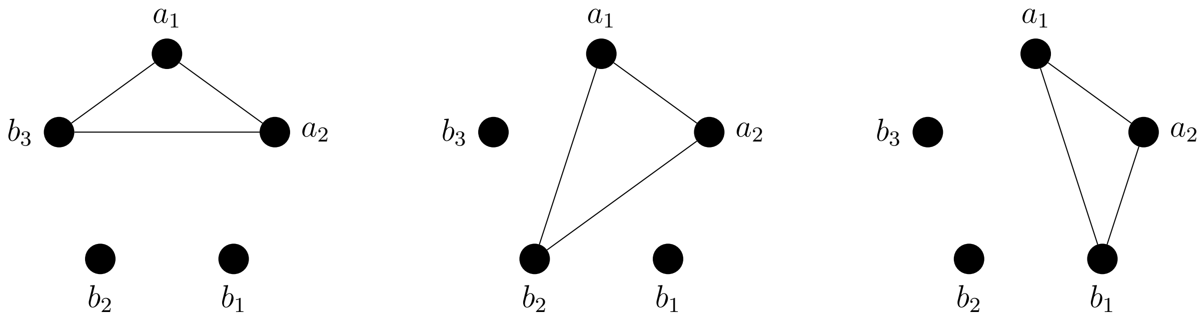


Figure 2.5: Colouring graphs for class 1A

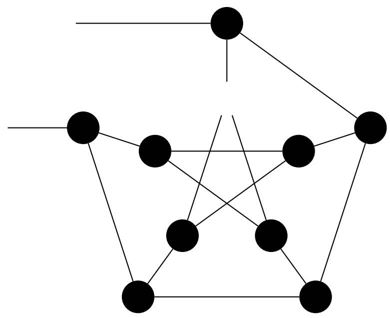


Figure 2.6: Example of a proper (2,3)-pole from class 1A

2.3.3 Class 2B

Unlike Class 1A, when two vertices are connected with a_1 and a_2, an edge does not have to be between them. The visualization is in Figure 2.7. An example of a proper (2,3)-pole from class 2B can be seen in Figure 2.8

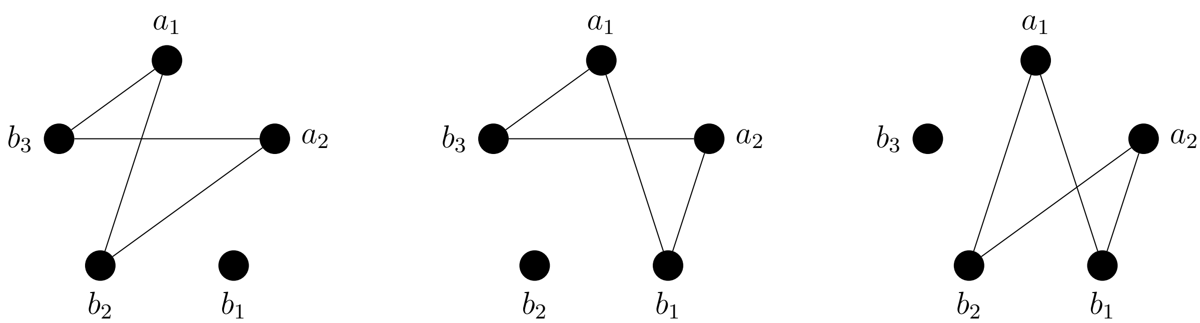


Figure 2.7: Colouring graphs for class 2B

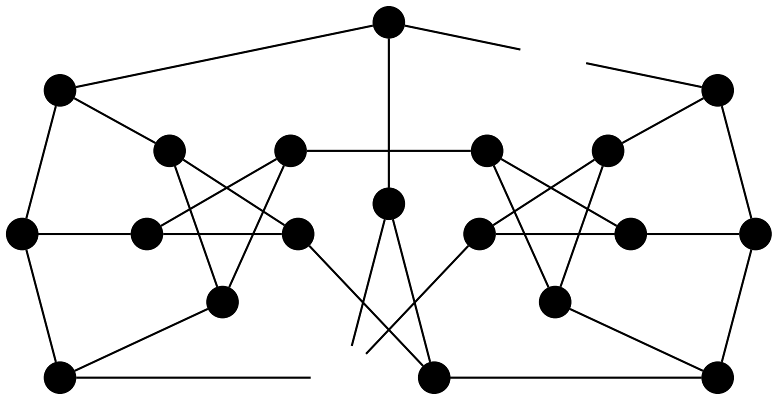


Figure 2.8: Example of a proper (2,3)-pole from class 2B

2.3.4 Class 2A

Since there are three pairs of vertices from b_1, b_2 and b_3, as in Class 1A, there are three possibilities. They are visualized in Figure 2.9. An example of a proper (2,3)-pole from class 2A can be seen in Figure 2.10.

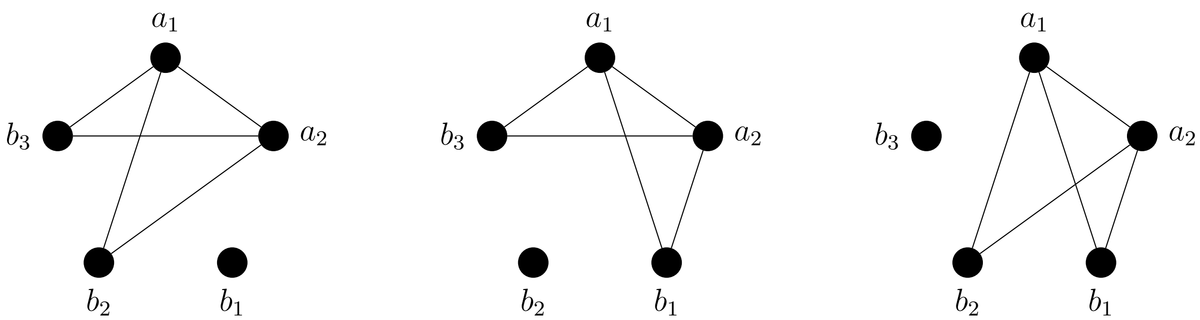


Figure 2.9: Colouring graphs for class 2A

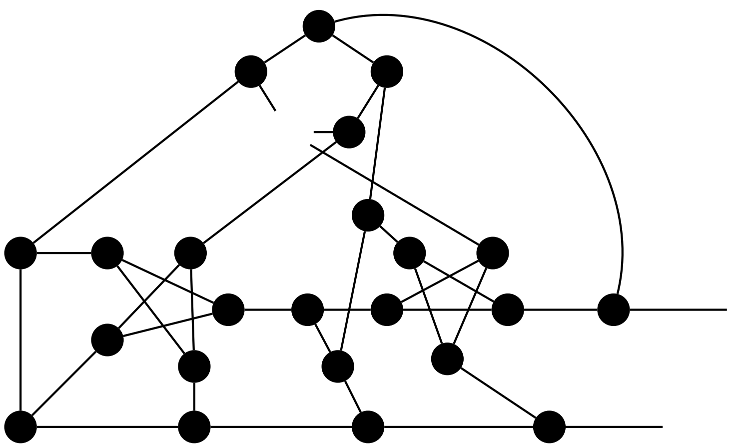
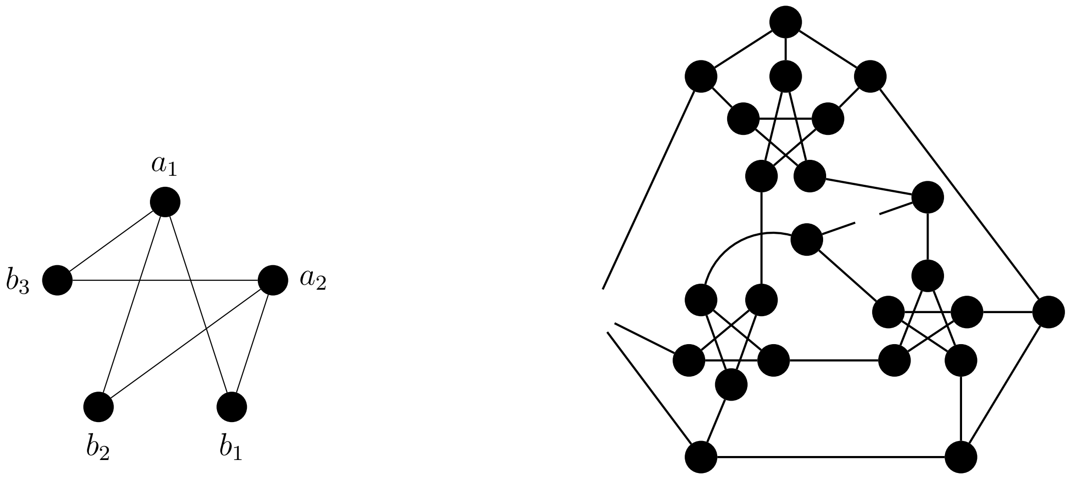


Figure 2.10: Example of a proper (2,3)-pole from class 2A

2.3.5 Class 3B

This class is almost the same as perfect; however, it does not allow a colouring where a_1 and a_2 are solitary. It is visualized in Figure 2.11a. An example of a proper (2,3)-pole from class 3B can be seen in Figure 2.11b.



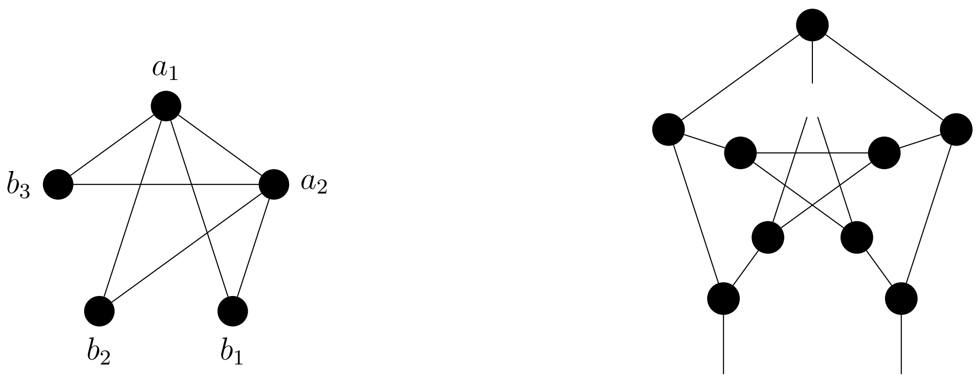
(a) Colouring graph for class 3B

(b) Example of a proper (2,3)-pole from class 3B

Figure 2.11: Proper (2,3)-poles from class 3B

2.3.6 Perfect

A proper (2,3)-pole is perfect if its colouring set coincides with C. This means its colouring set can be visualized as in Figure 2.12a. In our notation, this would be named class 3A. An example of a perfect proper (2,3)-pole can be seen in Figure 2.12b, resulting from the Petersen graph, where the distance between the removed vertex and the severed edge is more than 1. Each proper (2,3)-pole resulting from the Petersen graph is perfect if the distance is more than 1, otherwise, in class 1A.



(a) Perfect colouring of proper (2,3)-poles

(b) Example of a perfect proper (2,3)-pole

Figure 2.12: Perfect proper (2,3)-poles

Chapter 3

Our Work

3.1 Methods of Analysis

All of the results in this chapter come from our analysis conducted on several proper (2,3)-poles. For this reason, we have created a simple program in `C++` that helps us get the desired results. The logic behind representing graphs in the program and some basic operations on them is done by the `ba_graph` library [15]. As input, it receives a list of snarks in graph6 format [16], parses them, and performs the following operations on each.

Since the proper (2,3)-poles are multipoles resulting from a snark by removing one vertex and severing one edge, this is exactly what the program does: for each vertex v and edge e, where e is not incident with v, it removes v, severes e, and thus creates a proper (2,3)-pole. Thus, we have multiple proper (2,3)-poles from one snark.

Let T be the proper (2,3)-pole resulting from snark G, after removing the vertex v and severing the edge xy, $x \neq y \neq v \neq x$. We compute or observe the following properties for each proper (2,3)-pole:

- the resulting multipole in graph6 format;
- which edge and vertex were removed from the former snark;
- in which colouring class it is (see Section 2.3);
- the distance between the removed vertex and severed edge;
- how many pairs of vertices from $\{v, x\}, \{v, y\}$ are removable;
- how many pairs of edges $\{xy, va\}, \{xy, vb\}, \{xy, vc\}$ are removable, where a, b, c are neighbours of v.

For each graph on input, these results are then saved in a separate file containing a row for each proper (2,3)-pole originating from it.

The source code of the program can be found in the Appendix, or at `https://github.com/erehulka/proper-2-3-poles`.

3.2 Obtaining perfect proper (2,3)-poles

It may be convenient to modify some proper (2,3)-poles by adding some vertices and edges to obtain perfect proper (2,3)-pole. If we know in which colouring class the proper (2,3)-pole is, then we can make a junction with the specific constructions provided in this chapter to obtain perfect colouring, of course, only if the former (2,3)-pole is colourable.

3.2.1 Extending class 1A to 2B

Let $T(A, B)$ be a proper (2,3)-pole, whose colouring class is 1A and let it allow a solitary cycle $b_i a_1 a_2$ for some $b_i \in B$. Now let $T'(A, B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole M shown in Figure 3.1. In this partial junction we connect the connector B and the connector (c_1, c_2, c_3) such that it contains a junction of b_i and c_1. We prove that the result is a proper (2,3)-pole from colouring class 2B, which allows solitary cycle $b'_2 a_1 b'_3 a_2$.

Proof. Without loss of generality, let the semiedge b_i be b_1. Let this colouring of T be ϕ. As can be seen in Figure 3.2, using colourings of M ϕ_1 where $\phi_1(b'_1, b'_2, b'_3) = (3, 3, 2)$ and ϕ_2 where $\phi_2(b'_1, b'_2, b'_3) = (3, 2, 3)$, in both cases $\phi_1(c_1, c_2, c_3) = \phi_2(c_1, c_2, c_3) = (2, 1, 1)$. After the mentioned partial junction of T and M, we can colour the rest of T' by the colouring ϕ. We can see that the colouring graph of T' allows a solitary cycle $b'_2 a_1 b'_3 a_2$. No more colourings can be obtained since, as it can be seen, b'_2 and b'_3 must have different colours, so one of them is always solitary and we cannot obtain classes $2A, 3B$, and perfect. $\square$

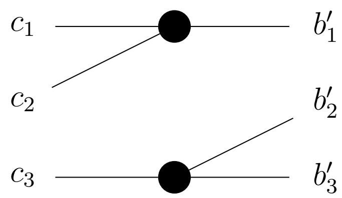


Figure 3.1: 6-pole used to create colouring class 2B from 1A

It is the smallest such 6-pole, considering the number of vertices, which extends class 1A to 2B. On zero vertices the only possibility are three isolated edges, which, as it can be checked, do not extend the colouring. It is not possible to get a 6-pole on one vertex, thus two vertices are the minimal amount.

Figure 3.2: Colourings of the 6-pole used to extend class 1A to 2B

3.2.2 Extending class 2B to 2A

Let $T(A, B)$ be a proper (2,3)-pole whose colouring class is 2B. Let $b_i, b_j, i \neq j$ be the two semiedges from B, for which there exists a solitary cycle $a_1 b_i a_2 b_j$. Now let $T'(A, B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole on Figure 3.3 by the junction of semiedges b_i to c_1, b_j to c_2 and the last semiedge to c_3. The result T' is from a colouring class $2A$ and allows solitary cycle $a_1 b'_1 a_2 b'_2 a_2$. The proof is similar to the one in extending class 1A to 2B.

Proof. Without loss of generality, let b_i, b_j be b_1, b_2 respectively. Let us denote the structure in Figure 3.3 as M. If we order the semiedges of M as $(c_1, c_2, c_3, b'_1, b'_2, b'_3)$, we see that its colouring set contains among others the colourings $(2, 1, 1, 2, 1, 1)$, $(1, 2, 1, 1, 2, 1)$, $(2, 1, 1, 2, 2, 2)$, thus after the mentioned junction we can colour the result in a way that the solitary cycle will be $a_1 b'_1 a_2 a_1 b'_2 a_2$.

Let's say we would want the solitary cycle of the result to contain b'_3 as well, in a colouring ϕ. This would mean that $\phi(b'_1) = \phi(b'_2)$, and both are not equal to $\phi(b'_3)$. However, as it can be checked, the colours of the resulting semiedges would be unambiguously set as $\phi(c_1) = \phi(c_2) = \phi(b'_1) = \phi(b'_2)$ and $\phi(c_3) = \phi(b'_3)$. Since we perform the junction as explained above, b_3 has to be contained in the solitary cycle of $T(A, B)$, which is false. This means that we can't obtain the class 3B nor the perfect colouring. □

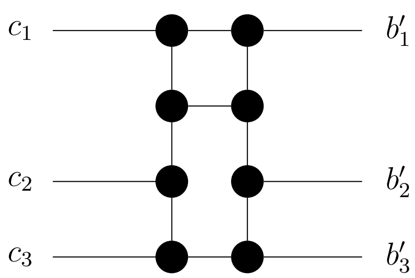


Figure 3.3: 6-pole used to extend multiple colouring classes

It must be noted that this construction produces proper (2,3)-poles which may not be contained in a nontrivial snark, since it contains a quadrilateral. If we would need to extend some proper (2,3)-pole to obtain a specific colouring class and require the extended proper (2,3)-pole to be contained in a nontrivial snark, we would need to use other, more complex constructions.

3.2.3 Extending proper (2,3)-poles from class 2A to perfect

To extend a proper (2,3)-pole from the colouring class 2A to a perfect one, the same 6-pole can be used as before, just with a different junction. Let $T(A,B)$ be a proper (2,3)-pole whose colouring class is 2B. Let $b_i, b_j, i \neq j$ be the two semiedges from B, for which there exists a solitary cycle $a_1 b_i a_2 a_1 b_j a_2$. Now let $T'(A,B')$, $B' = \{b'_1, b'_2, b'_3\}$ be a proper (2,3)-pole obtained by the junction of T with the 6-pole in Figure 3.3 by the junction of semiedges b_i to c_2, b_j to c_3 and the last semiedge to c_1. The result T' is a perfect proper (2,3)-pole, which can be proved similarly to before.

Proof. Let b_i, b_j be b_1, b_2 respectively, so in the junction we connect b_1 to c_2 and b_2 to c_3. Let us denote the structure in Figure 3.3 as M. If we order the semiedges of M as $(c_1, c_2, c_3, b'_1, b'_2, b'_3)$, we see that its colouring set contains the colourings $(2,1,2,1,2,2), (2,1,2,2,1,2), (2,2,1,2,2,1), (2,2,2,2,2,2)$, thus after the mentioned junction we can colour the result in a way that it will be perfect. □

3.2.4 Extending proper (2,3)-poles from class 3B to perfect

Let $T(A,B)$ be a proper (2,3)-pole whose colouring class is 3B. Let $T'(A,B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole on Figure 3.3, performing the junction of B to (c_1, c_2, c_3). Then T' is perfect, which can be proved similarly to before.

Proof. Let us denote the structure in Figure 3.3 as M. If we order the semiedges of M as $(c_1, c_2, c_3, b'_1, b'_2, b'_3)$, we see that its colouring set contains the colourings $(2,1,2,1,2,2), (2,1,2,2,1,2), (2,2,1,2,2,1), (2,1,1,2,2,2)$, thus after the mentioned junction we can colour the result in a way that it will be perfect. □

It is possible to incrementally modify each colourable proper (2,3)-pole to obtain a perfect one. For example, from class 1A it is possible to get class 2B, then 2A and finally perfect. It is evident that extending uncolourable multipoles to obtain colourable is impossible.

3.3 Theorems

Since when creating a proper (2,3)-pole from a snark, we are removing a vertex and severing an edge, we cannot look at the removable vertices *per se* since only one vertex is removed. However, we may look at the end vertices of the severed edge. By doing that, we discover a first proposition about the uncolourability of proper (2,3)-poles. At first, it may be helpful to define what a submultipole is.

Definition 14. Let M and N be multipoles. We say that M is a submultipole of N, denoted by $M \subseteq N$ if a multipole J exists such that N is a partial junction of M and J.

In other words, M is a submultipole of N if it can be extended to it by adding vertices, edges and semiedges and connecting them. The following lemma applies to the colourings of submultipoles; thus is essential when proving some propositions in this chapter.

Lemma 4. *Let M and N be multipoles such that $M \subseteq N$. If N is colourable, then M is colourable as well.*

Proof. Since $M \subseteq N$, so N is a result of the junction of M and some multipole J, there is an edge cut X splitting N into M and J. Let ϕ be the colouring of N. After removing the edge cut X, the exact colouring can be applied to colour M. □

This also means that if M is uncolourable, N is uncolourable as well.

Let M and N be multipoles, both constructed from a snark G. Since we often consider the intersection $E(M) \cap E(N)$, we clarify that:

- A link ab of G is included in the intersection if and only if it is present in both multipoles.
- A dangling edge originating from a vertex a which originated from an edge ab of G is included in the intersection if and only if it is present in both multipoles.

Proposition 1. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper (2,3)-pole $R(G; v; ab)$. If at least one of the pairs $\{v, a\}$ and $\{v, b\}$ is removable, then $T(A, B)$ is uncolourable.*

Proof. Let the removable pair be $\{v, a\}$, meaning that $R(G; \{v, a\}; \emptyset)$ is uncolourable. We see that $R(G; \{v, a\}; \emptyset) \subseteq T(A, B)$, so because of Lemma 4 the proper (2,3)-pole $T(A, B)$ is uncolourable. □

It must be noted, though, that the converse implication does not hold. There are several uncolourable proper (2,3)-poles, resulting from a snark, in which both of the pairs of vertices are unremovable. One of them is the mentioned example in Figure 2.4b.

Another interesting attribute in the question of colourability is the edge removability. Since the removed vertex in the creation of a proper $(2,3)$-pole has three neighbours, we can look at the removability of all three in pairs, along with the severed edge in the creation. One interesting proposition is also connected to the uncolourable proper (2,3)-poles.

Proposition 2. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A,B)$ a proper (2,3)-pole $R(G;v;ab)$. Let x,y,z be the neighbouring vertices of v in G. $T(A,B)$ is uncolourable if and only if all three pairs $\{ab,vx\},\{ab,vy\},\{ab,vz\}$ are removable.*

Proof. Suppose on the contrary that $T(A,B)$ is uncolourable and at least one pair from $\{ab,vx\},\{ab,vy\},\{ab,vz\}$ is unremovable, say $\{ab,vx\}$. This means that $R(G;\emptyset;\{ab,vx\})$ is colourable. However, $T(A,B)$ is a submultipole of $R(G;\emptyset;\{ab,vx\})$ and since $T(A,B)$ is uncolourable, $R(G;\emptyset;\{ab,vx\})$ must also be uncolourable because of Lemma 4, leading to a contradiction. Therefore, if $T(A,B)$ is uncolourable, all three pairs $\{ab,vx\},\{ab,vy\},\{ab,vz\}$ are removable.

Now for the proof of the second implication, suppose that all three edge pairs are removable and $T(A,B)$ is colourable. Let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z. By the definition of colouring classes, it is evident that $T(A,B)$ must allow a colouring, among others, where the solitary semiedges are a_1 and some semiedge b_i, say b_1. It is now possible to use this colouring, say ϕ, to colour $R(G;\emptyset;\{ab,vy\})$. Let us denote $R(G;\emptyset;\{ab,vy\})$ by R. We define a colouring ϕ' of R as follows: For each edge $e \in E(R) \cap E(T(A,B))$, the colour $\phi'(e)$ is equal to $\phi(e)$. The only edges not in this intersection are vx, vz and the dangling edge from v, let us denote it by d. We can set $\phi'(vx) = \phi(b_1)$, $\phi'(vz) = \phi(b_3)$. These two colours are different, since in ϕ the semiedge b_1 is solitary and b_3 sociable. Thus we can colour the last edge, d, with the colour different from $\phi'(vx)$ and $\phi'(vz)$. Since $\{ab,vy\}$ is removable, R is uncolourable, leading to a contradiction. □

Before the following proposition, we shall prove that for these pairs of edges, it cannot happen that exactly two are removable.

Lemma 5. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$. Let x,y,z be the neighbouring vertices of v in G. It is not possible that exactly two of the pairs $\{ab,vx\},\{ab,vy\},\{ab,vz\}$ are removable.*

Proof. We prove that if one of the pairs is unremovable, then at least one of the remaining pairs is unremovable as well. Let $\{ab,vx\}$ be unremovable, meaning that $R = R(G;\emptyset;\{ab,vx\})$ is colourable, let the colouring be ϕ. Let a_1, a_2 be the semiedges resulting from severing the edge ab and c_1, c_2 from severing vx, such that c_1 is part of the edge from x and c_2 of the edge from v. Because of the Parity Lemma and the fact that G is a snark, a_1 must have a different colour than a_2, and c_1, c_2 must be coloured with the same colours as them, also different from each other. The colours in ϕ of edges incident with v are all different, meaning that one of the edges, say vy, is coloured by the same colour as c_1. It cannot be the dangling edge, since $\phi(c_1) \neq \phi(c_2)$.

Now we can colour $R' = R(G; \emptyset; \{ab, vy\})$ with a colouring ϕ'. For each edge $e \in E(R) \cap E(R')$, $\phi'(e) = \phi(e)$. The only edges from R' not in this intersection are the edge vx, the dangling edge from y and the dangling edge from v. Let us denote the dangling edges by d, e, respectively. We will colour them the following way: $\phi'(vx) = \phi(c_1)$, $\phi'(d) = \phi(vy)$, $\phi'(e) = \phi(c_2)$. Since $\phi(c_1) \neq \phi(c_2)$, all three colours of edges incident with v in ϕ' will indeed be different. This means, that the pair $\{ab, vy\}$ is also unremovable.

The statement that exactly two of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are removable is equivalent to the statement that exactly one of them is unremovable. However, we have proved that this is impossible, since the presence of an unremovable pair implies the existence of another unremovable pair. □

Proposition 3. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper $(2,3)$-pole $R(G; v; ab)$. Let x, y, z be the neighbouring vertices of v in G. The proper (2,3)-pole $T(A, B)$ is from the class 1A if and only if exactly one of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ is removable.*

Proof. Suppose that $T(A, B)$ is from the class $1A$ and not exactly one of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ is removable. If all three pairs were removable, then by Proposition 2, $T(A, B)$ would be uncolourable. Also, there cannot be exactly two removable, as we have proved in Lemma 5. That means we can only explore the cases where none of the pairs is removable. Let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z.

Let $T(A, B)$ allow a solitary cycle $a_1b_1a_2$, implying it allows a colouring where the solitary semiedges are a_1 and b_1. Therefore $T(A, B)$ does not allow a colouring where one of the solitary semiedges is b_2 or b_3.

Let $R = R(G; \emptyset; \{ab, vx\})$ and let us denote the semiedges resulting from severing the edge vx by c_1, c_2, such that c_1 is part of the dangling edge from x and c_2 of the dangling edge from v. Since each pair is unremovable, a colouring ϕ of R exists. We see that $\phi(vy) \neq \phi(vz)$ and $T(A, B)$ is a submultipole of R. We can now construct a colouring ϕ' of $T(A, B)$ the following way. For each edge $e \in E(T(A, B)) \cap E(R)$, $\phi'(e) = \phi(e)$. The only edges from $T(A, B)$ not in this intersection are the dangling edges containing b_2 and b_3. We will colour them with $\phi'(b_2) = \phi(vy)$ and $\phi'(b_3) = \phi(vz)$. However, since $\phi(vy) \neq \phi(vz)$, the colour of b_2 is different from the colour of b_3, thus one of them is solitary in this colouring along with a_1 or a_2. This leads to a contradiction, since we suppose that $T(A, B)$ does not allow a colouring where one of the solitary semiedges is b_2 or b_3.

Now we can prove the second implication. Assume that exactly one of the pairs $\{ab, vx\}, \{ab, vy\}$,
$\{ab, vz\}$ is removable, let it be $\{ab, vx\}$, meaning that the pairs $\{ab, vy\}$ and $\{ab, vz\}$ are unremovable. As in the proof of the previous implication, let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z. Based on Proposition 2 $T(A, B)$ is colourable, so we can explore which semiedges from b_1, b_2, b_3 can be in its solitary cycle.

Suppose b_2 is in the solitary cycle of $T(A, B)$, implying the existence of a colouring ϕ_2 in which the solitary semiedges are a_1 and b_2. This would mean that $\phi_2(b_2) \neq \phi_2(b_1)$, $\phi_2(b_2) \neq \phi_2(b_3)$, $\phi_2(b_1) = \phi_2(b_3)$. From this colouring we can now construct a colouring ϕ_x of $R_x = R(G; \emptyset; \{ab, vx\})$: for each edge $e \in E(R_x) \cap E(T(A, B))$ the colour will be $\phi_x(e) = \phi_2(e)$. The only edges from R_x not included in the intersection are vy, vz and the dangling edge from v, let us denote it as d. We can then set $\phi_x(vy) = \phi_2(b_2), \phi_x(vz) = \phi_2(b_3)$ and $\phi_x(d)$ as the remaining colour, different from the two colours already set for vy and vz. Since $\{ab, vx\}$ is removable, the assumption that b_2 is in the solitary cycle leads to a contradiction.

Now suppose b_3 is in the solitary cycle of $T(A, B)$, implying the existence of a colouring ϕ_3 in which the solitary semiedges are a_1 and b_3. This would mean that $\phi_3(b_3) \neq \phi_3(b_1)$, $\phi_3(b_3) \neq \phi_3(b_2)$, $\phi_3(b_1) = \phi_3(b_2)$. From this colouring we can now also construct a colouring ϕ_x of R_x as before: for each edge $e \in E(R_x) \cap E(T(A, B))$ the colour will be $\phi_x(e) = \phi_3(e)$. The only edges from R_x not included in the intersection are vy, vz and the dangling edge from v, let us denote it as d. We can then set $\phi_x(vy) = \phi_3(b_2), \phi_x(vz) = \phi_3(b_3)$ and $\phi_x(d)$ as the other colour from the two colours already set for vy and vz. Since $\{ab, vx\}$ is removable, the assumption that b_3 is in the solitary cycle also leads to a contradiction.

Because $T(A, B)$ is colourable and as we have shown b_2 and b_3 cannot be in its solitary cycle, it must contain b_1, implying the existence of colouring where the solitary pairs are a_1, a_2; a_1, b_1; a_2, b_2; which coincides with the colouring class 1A. □

Based on this we can provide an interesting corollary for the other classes, implied by Proposition 2, Lemma 5 and Proposition 3.

Corollary 1. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper $(2, 3)$-pole $R(G; v; ab)$. Let x, y, z be the neighbouring vertices of v in G. $T(A, B)$ is perfect or from the class 2A, 2B or 3B, if and only if all three of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are unremovable.*

Proposition 4. *Let G be a snark, v its vertex, ab its edge where $a \neq v, b \neq v$ and the distance between ab and v is 1, that means a or b is a neighbour of v. Let $T(A, B)$*

be a proper $(2,3)$*-pole* $R(G;v;ab)$*. Then* $T(A,B)$ *is either uncolourable or its colouring set is from the class 1A.*

Proof. Since the distance between ab and v is 1, at least one of the vertices a, b is the neighbour of v; let it be a. Now there are two dangling edges from the vertex a; let their semiedges be a_1 and b_1. Because of this, in each colouring of $T(A,B)$, the colours of a_1 and b_1 are different. This means that $T(A,B)$ does not allow colourings where the solitary semiedges are a_2 with b_2, or a_2 with b_3. It is evident that the only possible colouring classes are 1A and uncolourable. □

Corollary 2. *Let* G *be a snark,* v *its vertex,* ab *its edge where* $a \neq v$ *and* $b \neq v$ *and* $T(A,B)$ *a proper* $(2,3)$*-pole* $R(G;v;ab)$*. Let* x, y, z *be the neighbouring vertices of* v *in* G*. If* $T(A,B)$ *is perfect, then all three pairs* $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ *are unremovable.*

Proof. If exactly one pair was removable, then by Proposition 3 $T(A,B)$ would be from class 1A. By Lemma 5 it is not possible that exactly two pairs are removable and by Proposition 2, if all three pairs were removable, $T(A,B)$ would be uncolourable. Thus all three pairs must be unremovable. □

The converse implication does not hold. An example can be seen in Figure 2.8, as the mentioned example of a proper (2,3)-pole from class 2B.

3.4 Data and Observations

To clarify how we got the propositions or how the data looks, we provide statistics about the explored snarks and their resulting proper (2,3)-poles. First, here is an example of the output table for the Petersen graph.

graph6	edge	vertex	colourings_class	distance	removable_vertices	removable_edges
MAMBHB@_?OA?@??O?	1, 5	0	perfect	2	0	0
MAkBHB@_?GA?@??O?	1, 6	0	1A	1	0	1
...						

As an input, we have used all snarks with a girth at least five and at most 28 vertices, which is 3,247 snarks. There are precisely 3,476,400 proper (2,3)-poles resulting from them. Most of these results are perfect proper (2,3)-poles. The proportions are in Table 3.1.

By analyzing proper (2,3)-poles, we found that 8.59% of them are uncolourable, while 91.41% are colourable. Based on Corollary 2 and its converse implication, we examined the distribution of colouring classes when all of the mentioned pairs are unremovable. This analysis led to the observations presented in Table 3.2. We see that most of the proper (2,3)-poles are perfect, but the numbers are also the same as in

class	percentage	total number
perfect	66.13%	2,299,022
1A	20.73%	720,660
uncolourable	8.59%	298,720
2B	3.2%	111,139
3B	0.68%	23,630
2A	0.67%	23,229

Table 3.1: Proportion of colouring classes in explored proper (2,3)-poles

Table 3.1. The equivalence between all proper (2,3)-poles from the classes perfect, 2B, 3B and 2A; and having all three pairs of edges unremovable is proved in Corollary 1.

class	percentage	total number
perfect	93.57%	2,299,022
2B	4.52%	111,139
3B	0.96%	23,630
2A	0.95%	23,229

Table 3.2: Proportion of colouring classes for all three unremovable pairs of edges

Another interesting observation is that no proper (2,3)-pole from the explored ones has precisely two of the mentioned pair of edges removable. We have proved this in Lemma 5. The proportions can be seen in Table 3.3

removable edges	percentage	total number
0	70.68%	2,457,020
1	20.73%	720,660
3	8.59%	298,720

Table 3.3: Proportion of number of removable edges

A snark is *critical* if every pair of its distinct adjacent vertices is unremovable. Similarly, a snark is *cocritical* if every pair of its distinct nonadjacent vertices is unremovable. If a snark is both critical and cocritical, then we say that G is *bicritical* [7]. In other words, a snark is bicritical if every pair of its distinct vertices is unremovable.

Among the 3,247 explored snarks, only five produce only colourable proper (2,3)-poles. One is the Petersen graph, then one with 20 vertices, two with 22 and one with 28 vertices. The ones with 20 and 28 vertices are the Isaacs snarks J_5 and J_7 respectively. The two snarks with 22 vertices are the Loupekine snarks. Because of

Proposition 1, each of the five mentioned snarks is bicritical.

Now we can take a look at the bicritical snarks. We have explored each nontrivial bicritical snark with at most 32 vertices – precisely 278 of them. There are 306,396 proper (2,3)-poles resulting from them. The proportions of their colouring classes is in Table 3.4.

class	percentage	total number
perfect	79.24%	242,784
1A	19.85%	60,830
uncolourable	0.59%	1,802
2A	0.32%	968
3B	0.00%	10
2B	0.00%	2

Table 3.4: Proportion of colouring classes in explored proper (2,3)-poles from bicritical snarks

As before, we can look at the proportions of the colouring classes, but only for the proper (2,3)-poles with all three of the mentioned edge pairs unremovable. The results are in Table 3.5. We see, that almost every such proper (2,3)-pole is perfect, however there is a small number of ones from the classes 2A, 3B, 2B. This may be an interesting observation for the further research about the sufficient conditions for a proper (2,3)-pole resulting from a bicritical snark to be perfect.

class	percentage	total number
perfect	99.6%	242,784
2A	0.4%	968
3B	0.00%	10
2B	0.00%	2

Table 3.5: Proportion of colouring classes for all three unremovable pairs of edges in proper (2,3)-poles from bicritical snarks

3.5 Problems

During the writing of this thesis, several problems arose. For some of them, we can provide an answer, others may be interesting problems for further research.

Claim 1. A proper (2,3)-pole constructed from a bicritical snark is not always perfect.

An counterexample from the class 1A, resulting from the Petersen graph by severing an edge and removing a vertex with a distance of 1 from the edge can be seen in Figure 2.6. Even if we require the distance between the severed edge and the removed vertex to be more than 1 (since the results are always uncolourable or from class 1A, as proved in Proposition 4), we can see an uncolourable counterexample resulting from the second Blanuša snark in Figure 2.4b.

Claim 2. All proper (2,3)-poles resulting from the Double Star snark are perfect, when the distance between the removed vertex and the severed edge is more than one.

If the distance between the removed vertex and the severed edge is more than 1, the resulting proper (2,3)-pole is perfect. Otherwise, it is in the colouring class 1A. Precisely, 1080 proper (2,3)-poles are perfect and 180 are from the colouring class 1A.

Problem 1. Suppose we only consider multipoles resulting from snarks by severing an edge and removing a vertex with a distance of more than 1, since adjacent edges are trivially removable. Is a proper (2,3)-pole constructed from a snark without any removable pair of edges always perfect?

There are only four such snarks from the ones we have explored: the Petersen graph, the Isaacs snarks J_5 and J_7, and the Double Star snark. All of the proper (2,3)-poles resulting from these graphs, with the distance between the removed vertex and the severed edge more than 1 are indeed perfect. However we have not proved this statement, thus it can be explored in further research.

Problem 2. Construct multipoles used to extend colourings of proper (2,3)-poles, which allow the resulting proper (2,3)-poles to be contained in a nontrivial snark.

As mentioned in Section 3.2, one of the 6-poles contains a quadrilateral, so each snark of which it is a part of is trivial.

Problem 3. Construct an infinite family of snarks, that produce only colourable proper (2,3)-poles.

We have found several snarks producing only colourable proper (2,3)-poles: the Petersen graph, the Isaacs snarks J_5 and J_7 and the two Loupekine snarks of order 22. This may be helpful when exploring infinite families of snarks producing only colourable proper (2,3)-poles.

Problem 4. If we construct a proper (2,3)-pole from a bicritical snark in such a way, that the distance between the severed edge and the removed vertex is more than one, and both are a part of a 5-cycle (not necessarily the same), is the result always perfect?

If a counterexample is found, an additional requirement of $z(G) = 5$ for the snark could be imposed.

Conclusion

Proper (2,3)-poles are multipoles resulting from snarks by removing one vertex and severing an edge, not incident with the removed vertex. In our work, we analysed how the colourings of proper (2,3)-poles can look. Based on their colouring sets, we have divided them into six classes. For each of them, we have found an example, thus proving that each class can be obtainable.

One of the main goals was to explore and describe perfect proper (2,3)-poles. That is why we have also provided some constructions, specifically 6-poles, which, when performing a junction with some colourable proper (2,3)-pole from some colouring class, result in a new proper (2,3)-pole that is from another colouring class. This way, it is possible to incrementally use these constructions to get perfect proper (2,3)-poles from any colouring class.

We explored all snarks with girth at least five and at most 28 vertices. This amounts to a total of 3247 snarks. We also examined all proper (2,3)-poles resulting from them, which is precisely 3476400 of them. Based on our exploration, we formulated some propositions regarding the colouring of proper (2,3)-poles. These propositions provide the necessary and sufficient conditions for their specific colouring properties. Most of the propositions were about the question of colourability of the proper (2,3)-poles, but also about being part of some specific colouring class. In most cases, these propositions were connected to removable pairs of vertices or edges.

By looking at the data, we see that more than half of the explored proper (2,3)-poles are perfect. However, we have not fulfilled the goal of finding sufficient conditions for the proper (2,3)-pole to be perfect. Only one necessary condition was found and proved. This may be an interesting goal for further research; it may be possible to find sufficient conditions using the provided data, observations and propositions.

Also, we have explored all bicritical snarks with girth at least five and at most 32 vertices. This amounts to a total of 278 snarks, with 306396 proper (2,3)-poles resulting from them.

Another interesting question for further research may be finding snarks, which produce only colourable proper (2,3)-poles. In the explored snarks, we have found five of them. Based on our propositions, this problem may be connected to finding snarks which have most of their edge pairs or all of them unremovable.

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Appendix

We attach a CD with detailed results of our analysis and used computer program. The link to the computer program source code is
`https://github.com/erehulka/proper-2-3-poles`.