Channel assignment problem with variable weights

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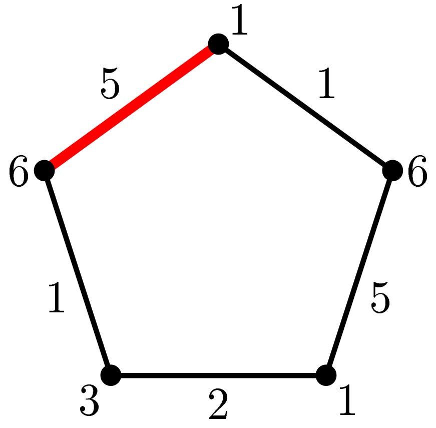
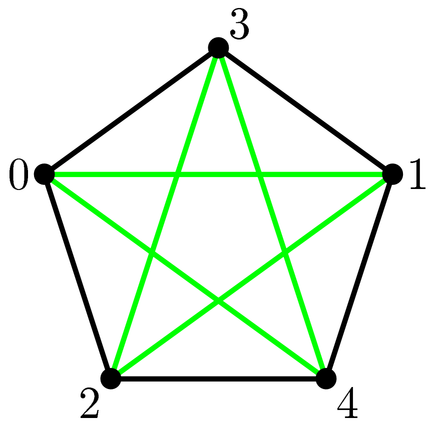
Channel assignment problem

- instance of the channel assignment problem
  a graph G with edge weights $w : E(G) \to \mathbb{N}$
- the goal of the problem
  a vertex labeling $c : V(G) \to \mathbb{N}$ such that $|c(u) - c(v)| \geq w(uv)$ for $uv \in E(G)$
- the span $\chi_w(G)$ is the minimum of $\max_{u \in V(G)} c(u)$ taken over all valid labelings c
  minimization of the difference between the maximum and the minimum label
- a generalization of graph coloring
  if $w(e) = 1$ for every $e \in E(G)$, then the span is equal to $\chi(G)$



Distance constrained labelings

- $L(p,q)$-labeling $c : V(G) \to \mathbb{N} \cup \{0\}$ of a graph G
  $|c(u) - c(v)| \geq p$ if u and v are adjacent (at distance one)
  $|c(u) - c(v)| \geq q$ if u and v have a common neighbor
- the span $\lambda_{p,q}(G)$ is the minimum of $\max\limits_{u \in V(G)} c(u)$ taken over all valid labelings c
- $L(p,q)$-labeling of G is a special case of the channel assignment problem for G^2
  the original edges of G have weights p, the new ones q



- $L(p_1, \ldots, p_k)$-labeling
  $|c(u) - c(v)| \geq p_i$ if the distance between u and v is i

$L(p_1, \ldots, p_k)$-labelings with variable weights

- first studied by Griggs and Jin (2004) but similar notions existed before
- $\lambda_G(p_1, \ldots, p_k)$ as function of $p_1, \ldots, p_k$ for a fixed graph G
- scaling property: $\alpha\lambda_G(p_1, \ldots, p_k) = \lambda_G(\alpha p_1, \ldots, \alpha p_k)$
- $p_1, \ldots, p_k$ are now allowed to be any non-negative reals
- G is allowed to be an infinite graph (planar triangular or hexagonal grids)
- they verified their conjectures for finite graphs and infinite graphs with $k = 2$

Piecewise Linearity Conjecture

For any graph G, the graph of the function $\lambda_G(p_1, \ldots, p_k)$ is comprised by finitely many linear parts, i.e., there exist finitely many hyperplanes in $\mathbb{R}^k$ through the origin of $\mathbb{R}^k$ such that the function $\lambda_G(p_1, \ldots, p_k)$ is linear in each of the convex polyhedral cones (formed by non-negative reals) that are determined by the hyperplanes.

Conjectures of Griggs and Jin (contd.)

Coefficient Bound Conjecture

For every graph G and every integer k, there exists a constant $D_{k,G}$ such that the following holds for all reals $p_1, \ldots, p_k$: the value of the function $\lambda_G(p_1, \ldots, p_k)$ is equal to $\sum_{i=1}^{k} \alpha_i p_i$ for some integer coefficients $\alpha_1, \ldots, \alpha_k$ between 0 and $D_{k,G}$ (the integers $\alpha_1, \ldots, \alpha_k$ may depend on $p_1, \ldots, p_k$). Moreover, there is a labeling c with span $\lambda_G(p_1, \ldots, p_k)$ such that $c(v) = \sum_{i=1}^{k} \alpha_i(v) p_i$ where $\alpha_1(v), \ldots, \alpha_k(v)$ are integers between 0 and $D_{k,G}$.

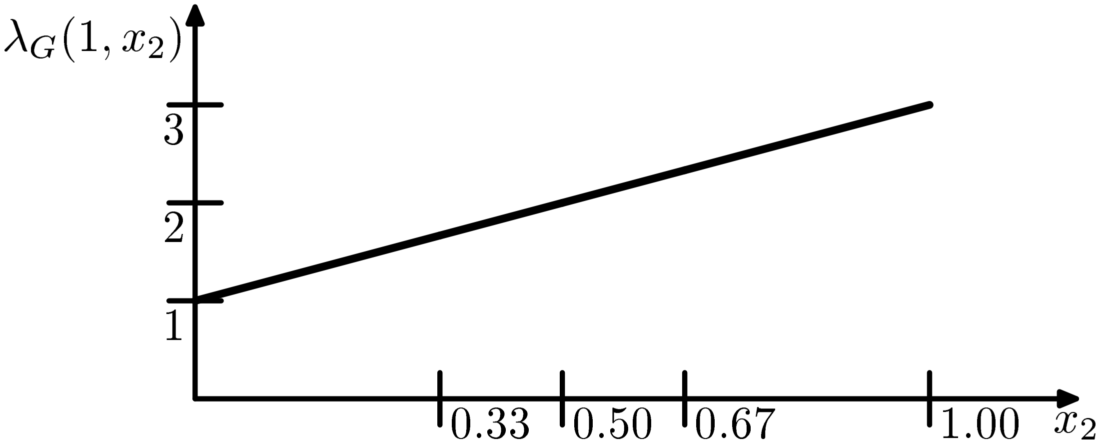
Delta Bound Conjecture

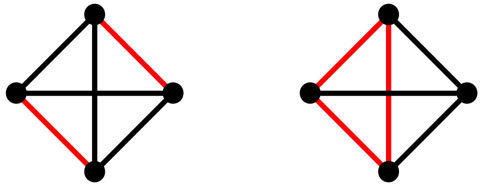
For every integers Δ and k, there exists a constant $D_{k,\Delta}$ such that, for every graph G with maximum degree at most Δ and every reals $p_1, \ldots, p_k$ the following holds: the value of the function $\lambda_G(p_1, \ldots, p_k)$ is equal to $\sum_{i=1}^{k} \alpha_i p_i$ for some integer coefficients $\alpha_1, \ldots, \alpha_k$ between 0 and $D_{k,\Delta}$ (the integers $\alpha_1, \ldots, \alpha_k$ may depend on $p_1, \ldots, p_k$). Moreover, there is a labeling c with span $\lambda_G(p_1, \ldots, p_k)$ such that $c(v) = \sum_{i=1}^{k} \alpha_i(v) p_i$ where $\alpha_1(v), \ldots, \alpha_k(v)$ are integers between 0 and $D_{k,\Delta}$.

λ-GRAPHS

- λ-graph is a graph with k types of edges
  x_1-edges, ..., x_k-edges
- two vertices may be joined by edges of several types
- a vertex labeling $c: V(G) \to \mathbb{R}$ such that $|c(u) - c(v)| \geq x_i$ for every x_i-edge uv
- the span $\lambda_G(x_1, \ldots, x_k)$ is the minimum of $\max\limits_{u \in V(G)} c(u)$ taken over all labelings c
  if G is infinite, maximum is replaced by supremum
- the minimum is attained even if G is infinite
- generalization of the notion studied by Jin and Griggs:
  vertices at distance i are joined by an x_i-edge

Examples (1/4)

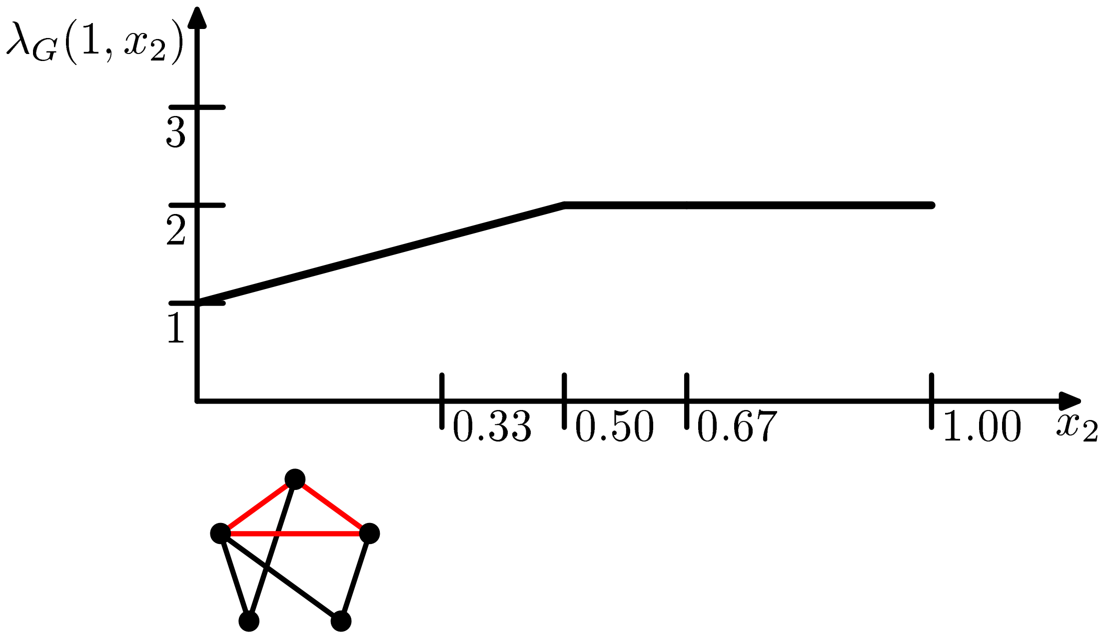




black x_1-edges, red x_2-edges

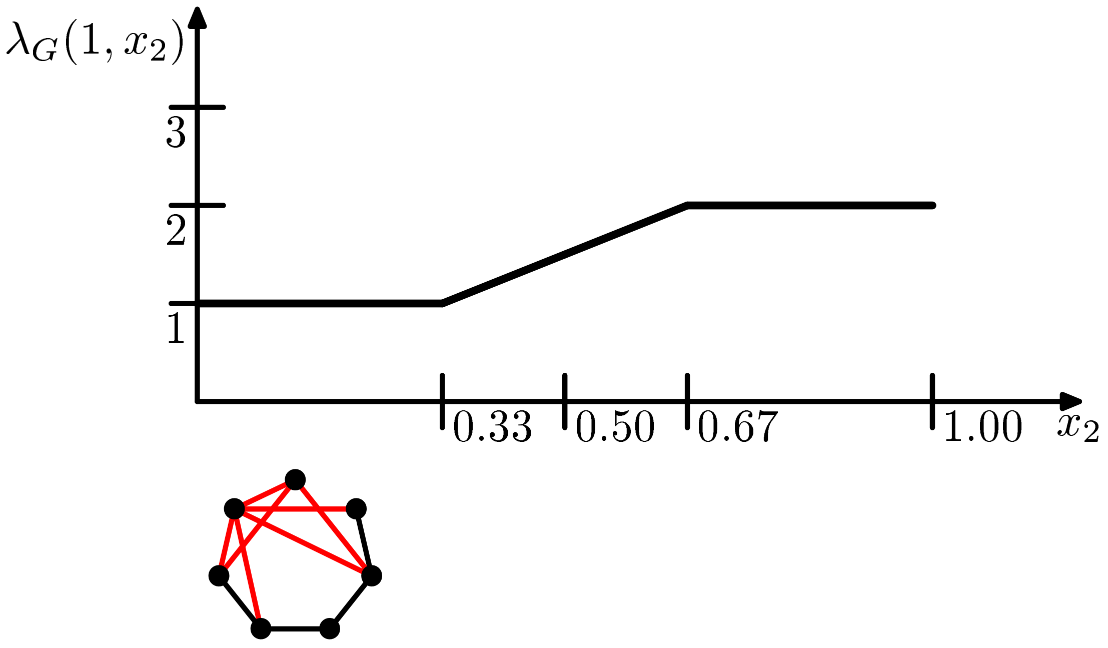
- because of the scaling property, we may assume that $x_1 = 1$

Examples (2/4)



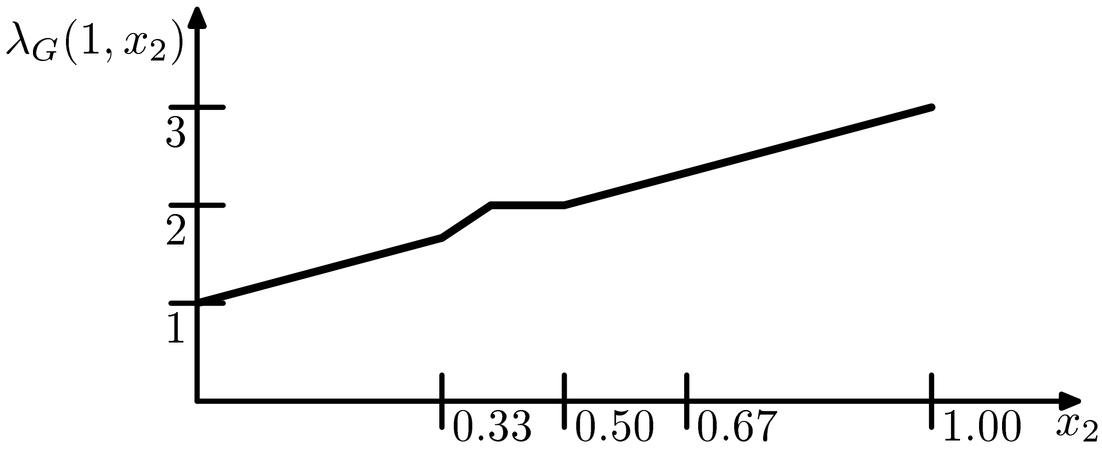
black x_1-edges, red x_2-edges

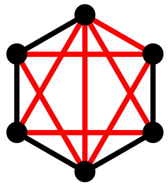
Examples (3/4)



black x_1-edges, red x_2-edges

Examples (4/4)





black x_1-edges, red x_2-edges

Main result

For every λ-graph G with k types of edges and chromatic number at most Λ, $\lambda_G(x_1, \ldots, x_k)$ is a piecewise linear function of $x_1, \ldots, x_k$ with at most $C_{k,\Lambda}$ linear parts formed by linear functions with integer coefficients between 0 and $D_{k,\Lambda}$.

Moreover, the subset of $\mathbb{R}^k$ formed by non-negative integers can be partitioned into at most $C_{k,\Lambda}$ (closed) polyhedral cones such that for each of the cones the following holds: there exist integers $\alpha_i(v)$ between 0 and $D_{k,\Lambda}$ such that the labeling c, $c(v) = \sum_{i=1}^{k} \alpha_i(v) x_i$, is a proper labeling of G with respect to $x_1, \ldots, x_k$ (within the cone) and the span of c is $\lambda_G(x_1, \ldots, x_k)$.

Proof technique

- an analogue of Gallai-Roy theorem for λ-graphs
- an orientation of a graph is finitary if there is a constant $K \geq 0$ such that every oriented walk has length at most K
- the weight of a path is the sum of weights of its edges
- the weight of an orientation is the maximum weight of an oriented path
- $\lambda_G(p_1, \ldots, p_k)$ is equal to the minimum weight of a finitary orientation of G

- we show that for any k-tuple $p_1, \ldots, p_k$ the maximum length of an oriented walk in an optimal finitary orientation is bounded by a constant $D_{k,\Lambda}$

Proof technique (contd.)

- sketch of ideas for $k = 2$, the general case is similar but more involved
- $\lambda_G(x_1, x_2) \le \Lambda \cdot \max\{x_1, x_2\}$
- if x_1 and x_2 are close, i.e., $\frac{\max\{x_1,x_2\}}{\min\{x_1,x_2\}}$ is bounded, then the length of an oriented walk in any optimum orientation does not exceed $\Lambda \cdot \frac{\max\{x_1,x_2\}}{\min\{x_1,x_2\}} = \text{const.}$
- the tricky case is that $x_1 \ll x_2$ (or $x_2 \ll x_1$)
- $\lambda_G(x_1, x_2) \approx \chi(G_2) \cdot x_2$ where G_2 is the subgraph induced by x_2-edges
- fix an optimum labeling c and partitiong $V(G)$ into parts $V_0, \ldots, V_{\chi(G_2)}$ with $c(v) \approx 0$, $c(v) \approx x_2$, $c(v) \approx 2x_2$, etc.
- replace the orientation in each of these parts by an orientation witnessing that $\chi(G) = \Lambda$ without affecting the maximum color used

Final remarks

How fast do the constants grow depending on k and Λ?

- the constants are extremely fast growing

$$D_{1,\Lambda} = \Lambda$$
$$D_{2,\Lambda} = 2^{(\Lambda+1)(2^{2\Lambda}+3)} \cdot \Lambda \approx 2^{2^{2\Lambda}}$$
$$D_{3,\Lambda} \approx 3^{3^{2^{2^{2\Lambda}}}}$$

  $C_{k,\Lambda}$ grows similarly

- we are not aware of any lower bound not polynomial in k and Λ