Antibandwidth and Cyclic Antibandwidth of Meshes and Hypercubes

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Dedicated to memory of Ondrej Sýkora

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Antibandwidth problem

- Consists of placing the vertices of a graph on a line in consecutive integer points in such a way that the minimum difference of adjacent vertices is maximized.
- Just another labeling problem (from graph theory point of view)

Confusing terminology

- Originally studied under the term ***separation number*** (Leung, Vornberger, *On some variants of the bandwidth minimization problem*)
- ***Dual bandwidth*** (Lin, Yuan)
- We propose (best and hopefully final) term: ***antibandwidth***

Previous results

- NP-complete (Leung, Vornberger)
- Polynomially solvable for the complements of
  - Interval
  - Arborescent comparability
  - Treshold

  graphs.

  (Donnely, Isaak, *Hamiltonian powers in … graphs)*

Previous results

- Exact results for:
  - Paths, cycles, special trees, complete and complete bipartite graphs

- Also interesting for disconnected graphs.
  - Exact values for graphs consisting of copies of simple graphs.

Previous results

- m x n mesh, $m \geq n$, (Miller, Pritikin, *On the separation number of graphs*)

$$\left\lceil \frac{n(m-1)}{2} \right\rceil \leq ab(P_m \times P_n) \leq \left\lfloor \frac{mn}{2} \right\rfloor$$

- N-dimensional hypercube (Miller, Pritikin,…)

$$2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}}(1+o(1)) \leq ab(Q_n) \leq 2^{n-1}$$

Our contribution

- Upper bound method suitable for bipartite graphs
- Improving bounds for hypercubes and meshes:

$$ab(P_m \times P_n) = \left\lceil \frac{n(m-1)}{2} \right\rceil$$

$$ab(Q_n) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}}(1 + o(1))$$

Our contribution

- Toroidal meshes $C_n \times C_n$:
  - Even n: $$ab(C_n \times C_n) = \frac{(n-2)n}{2}$$
  - Odd n: $$ab(C_n \times C_n) = \frac{(n-2)(n+1)}{2}$$

Meshes: upper bound

- *Definition:* Let $V_{1,} V_2$ *be a bipartition.* Minimal vertex boundary of a set $A \subseteq V_1$ is a set of all vertices from V_2 having neighbour in *A*.
- Proof based on result of Bezrukov and Piotrowski (minimal bipartite vertex boundary of mesh)

Meshes: lower bound

- Showed in Miller, Pritikin, *On separation number of graphs*

17	7	22	11	25
3	18	8	23	12
14	4	19	9	24
1	15	5	20	10
13	2	16	6	21

$$ab(P_m \times P_n) = \lceil \frac{(m-1)n}{2} \rceil$$

Even torus

- Optimal numbering for even torus $C_n \times C_n$

35	11	51	27	63	23	47	7
3	43	19	59	31	55	15	39
34	10	50	26	62	22	46	6
2	42	18	58	30	54	14	38
33	9	49	25	61	21	45	5
1	41	17	57	29	53	13	37
32	8	48	24	60	20	44	4
0	40	16	56	28	52	12	36

$$ab(C_n \times C_n) = \frac{n(n-2)}{2}$$

Odd torus

- Optimal numbering of odd torus $C_n \times C_n$

27	48	20	41	13	34	6
47	19	40	12	33	5	26
18	39	11	32	4	25	46
38	10	31	3	24	45	17
9	30	2	23	44	16	37
29	1	22	43	15	36	8
0	21	42	14	35	7	28

$$ab(C_n \times C_n) = \frac{(n-2)(n+1)}{2}$$

Hypercube

- Vertices of Q_n can be partitioned into sets $X_i, i=0,1,2,...,n$ according to their distance from the vertex 00...0.
- Edges are only between X_i and X_{i+1}.

$$ab(Q_n)=2^{n-1}-\frac{2^{n-1}}{\sqrt{2\pi n}}(1+o(1))$$

Cyclic antibandwidth

- The vertices are mapped bijectively into $C_{|V|}$ such that the minimal distance, measured in cycle, of adjacent vertices is maximized.
- We provide:
  - General lower bound
  - Values for meshes, tori and hypercubes

General bounds

- Upper bound

$$cab(G) \le ab(G)$$

- Lower bound

$$cab(G) \ge min_f(ab(G, f), mn - max_{(u,v)\in E}|f(u) - f(v)|)$$

Cyclic antibandw. of meshes

- m even, n odd, then

$$\lfloor \frac{n(m-1)}{2} \rfloor \leq cab(P_m \times P_n) \leq \lceil \frac{n(m-1)}{2} \rceil$$

- Otherwise

$$cab(P_m \times P_n) = \frac{n(m-1)}{2}$$

Cyclic antibandw. of meshes

- Another optimal numbering of mesh comparing the antibandwidth part

2	**14**	**6**	**18**	**10**
12	**4**	**16**	**8**	**20**
1	**13**	**5**	**17**	**9**
11	**3**	**15**	**7**	**19**

Cyclic antibandw. of meshes

- Way of proof
  - To show that previous numbering is also ab-optimal
  - Computing the length of the longest edge in this labeling
  - Getting the lower bound value from general formula

Cyclic antibandw. of tori

- Even torus

$$cab(C_n \times C_n) = ab(C_n \times C_n) = \frac{n(n-2)}{2}$$

- Odd torus

$$cab(C_n \times C_n) = ab(C_n \times C_n) = \frac{(n-2)(n+1)}{2}$$

Cyclic antibandw. of hypercube

- Similar way of proof as for mesh

$$cab(Q_n) = 2^{n-1} - \frac{2^{n-1}}{\sqrt{2\pi n}}(1 + o(1))$$

Conclusion

- Antibandwidth
  - Improved bounds for meshes and hypercubes
  - Results for toroidal meshes
- Cyclic antibandwidth
  - General bounds based on antibandwidth
  - Results for meshes, hypercubes and toroidal meshes

The End

Thank You