Applications of Superposition in Graph Thery

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Nowhere-zero flows in graphs

5-flow conjecture

3-flow conjecture

snarks

Cycles in graphs

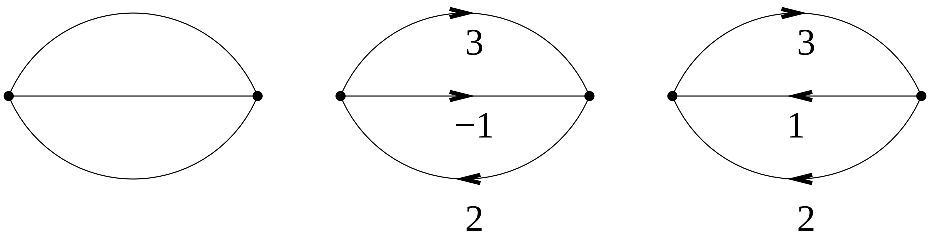
cycle double covering

dominating circuit conjecture

compatibility conjecture

Nowhere-zero flows

Graph G has a **nowhere-zero k-flow** if its edges can be oriented and assigned numbers $\pm 1, \ldots, \pm(k-1)$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph.



nowhere-zero 4-flow

$$G \text{ has a n.-z. } k\text{-flow} \Longrightarrow G \text{ has a n.-z. } (k+1)\text{-flow}$$

If A is an additive Abelian group, then G has a **nowhere-zero** **A-flow** if its edges can be oriented and assigned elements of $A - \{0\}$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of G.

Theorem (Tutte 1950, 1954): *Let G be a graph. Then the following statements are pairwise equivalent.*

(1) *G has a nowhere-zero k-flow.*

(2) *G has a nowhere-zero $\mathbb{Z}_k$-flow.*

(3) *G has a nowhere-zero A-flow for any $|A| = k$.*

Theorem (Tutte 1954): *A planar graph is k-colorable $\Longleftrightarrow$ its dual has a nowhere-zero k-flow.*

A cubic graph G has a n.-z. 4-flow $\Longleftrightarrow$ G is 3-edge-colorable

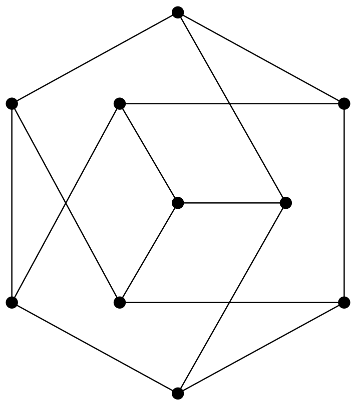
Theorem (Tait 1880, Tutte 1954): The following statements are equivalent.

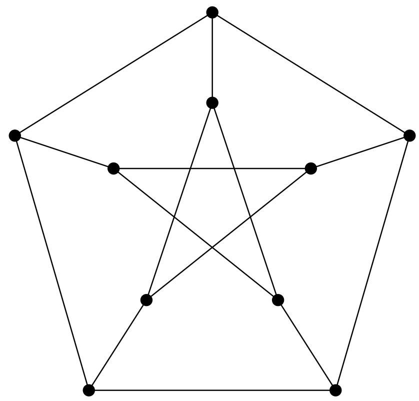
(1) Every planar graph is 4-colorable.

(2) Every bridgeless planar cubic graph is 3-edge-colorable.

(3) Every bridgeless planar graph has a nowhere-zero 4-flow.

Petersen graph has no 3-edge-coloring and no n.-z. 4-flow.





4-Flow Conjecture (Tutte 1966): Every bridgeless graph without a Petersen minor has a nowhere-zero 4-flow.

Theorem (Robertson, Sanders, Seymour, Thomas): $\forall$ *bridgeless cubic graphs without a Petersen minor has a n.-z.* 4*-flow.*

Theorem (Heawood 1890): *Every planar graph is* 5*-colorable.*

5-Flow Conjecture (Tutte 1954): Every bridgeless graph has a nowhere-zero 5-flow.

Theorem (Jaeger 1976, Kilpatrick 1975): *Every bridgeless graph has a nowhere-zero* 8*-flow.*

Theorem (Seymour 1981): *Every bridgeless graph has a nowhere-zero* 6*-flow.*

k**-snarks** = graphs without nowhere-zero k-flows

A graph with a bridge is a k-snark $\forall\, k \geq 2$.

snarks = cubic graphs - without nowhere-zero 4-flows

- cyclical edge-connectivity ≥ 4
- girth ≥ 5

A graph is **cyclically** k**-edge-connected** if deleting fewer than k edges does not disconnect the graph into two components having circuits.

CDC Conjecture (Seymour 1978, Szekeres 1973): Every bridgeless graph has a family of circuits which together cover each edge twice.

Proposition: *Smallest counterexamples to the* 5*-flow and CDC conjectures must be snarks.*

History of constructions of snarks:

Petersen graph (19th century) - 10 vertices

Blanuša (1946) - 18 vertices

Descartes [Tutte] (1948) - 210 vertices

Szekeres (1972) - 50 vertices

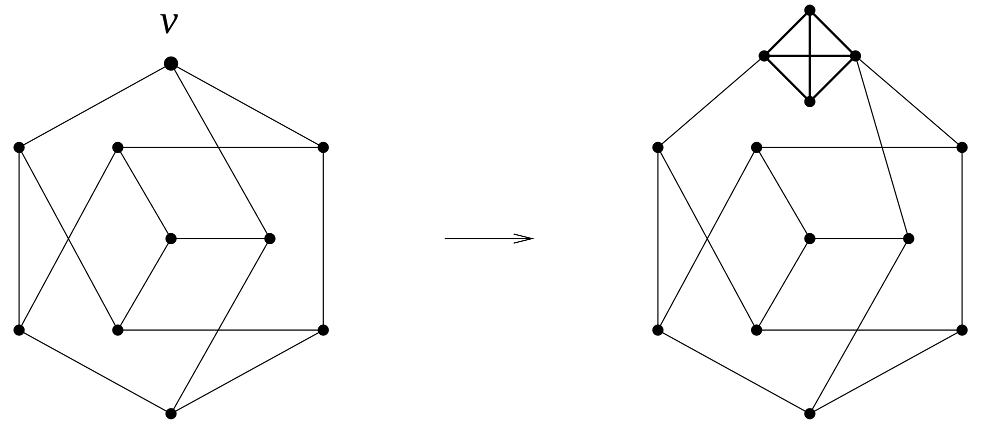
Infinite families:

BDS class: Isaacs (1975), Adelson-Velskij, Titov (1974)

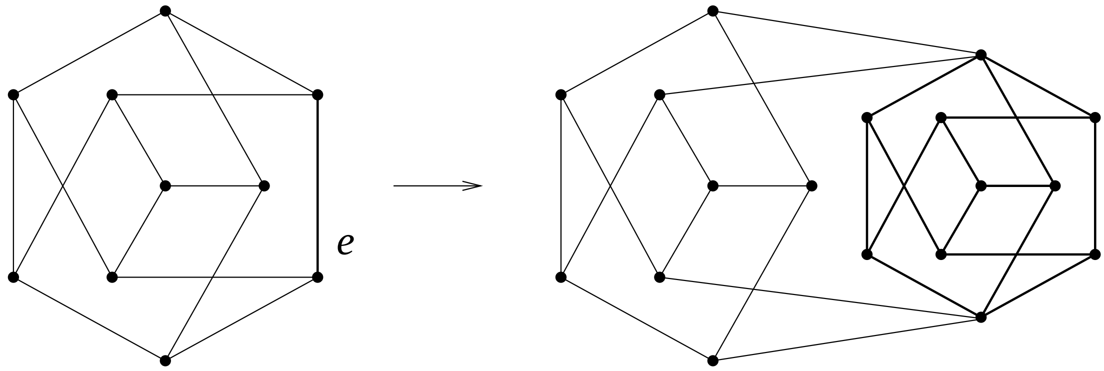
Flower snarks: Isaacs (1975), Grinberg

Theorem (Holyer 1981): *It is an NP-complete problem to decide whether a cubic graph is* 3*-edge-colorable.*

vertex superposition: replace $v \in V(G)$ by $\mathcal{S}(v)$



edge superposition: replace $e \in E(G)$ by $\mathcal{S}(e)$



an edge superposition is k**-proper** if $\mathcal{S}(e)$ is a k-snark

G' is a (k**-proper**) **superposition** of G if it arise after applying finitely many vertex and (k-proper) edge superpositions.

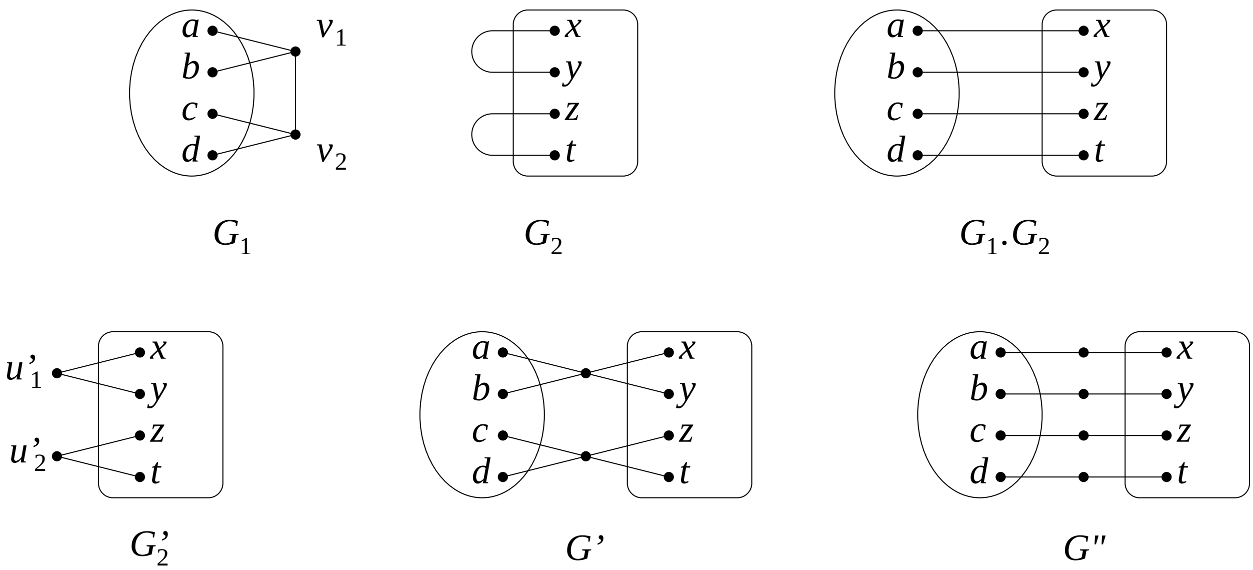
Theorem (K. 2002): *Every k-proper superposition of a k-snark is a k-snark*

Proposition: *If graphs H and G are homeomorphic, then*

$$H \text{ is a } k\text{-snark} \Longleftrightarrow G \text{ is a } k\text{-snark.}$$

Dot product

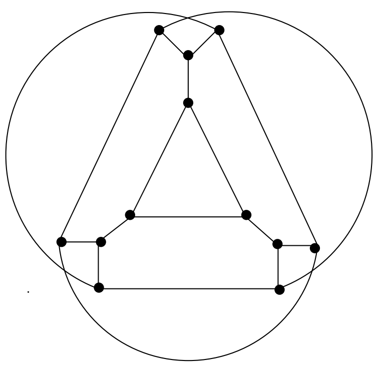
Isaacs (1975), Adelson-Velskij, Titov (1974) → BDS class



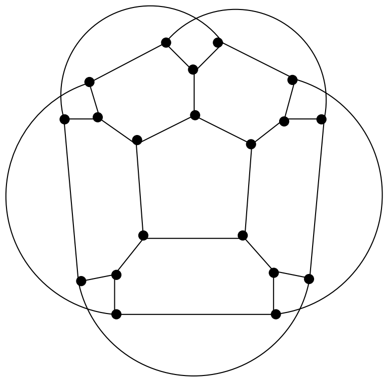
Cyclically 6-edge-connected snarks

Isaacs (1975), Grinberg → flower snarks of orders

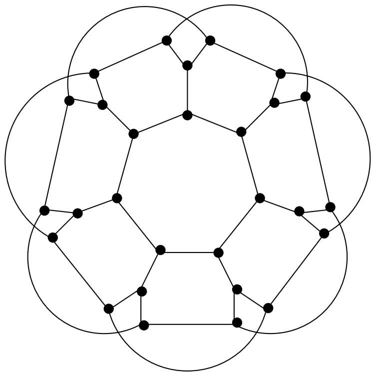
$$8k + 28,\ k \geq 0$$



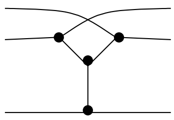
I_3

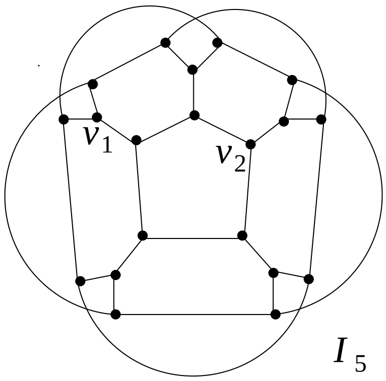


I_5

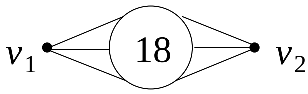


I_7

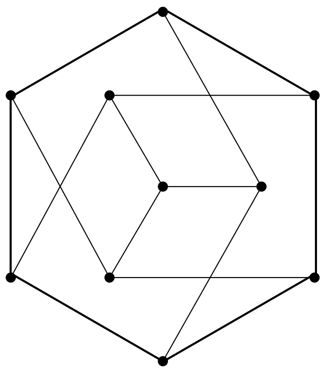


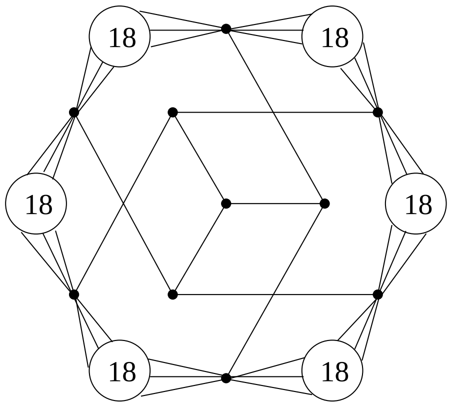


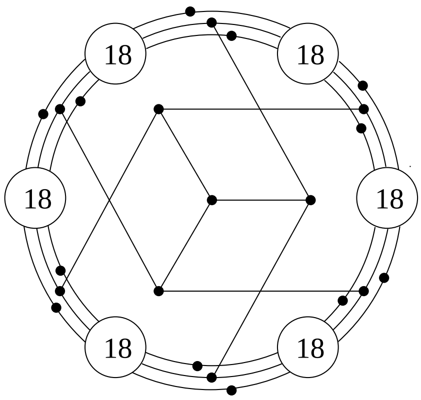
I_5

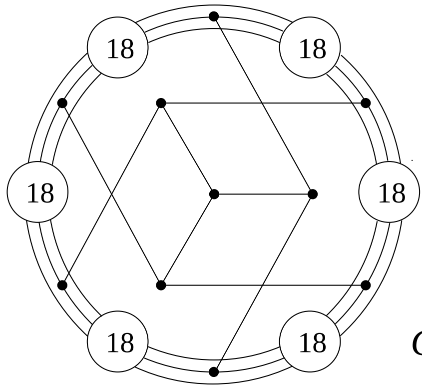


a symbolic representation of I_5

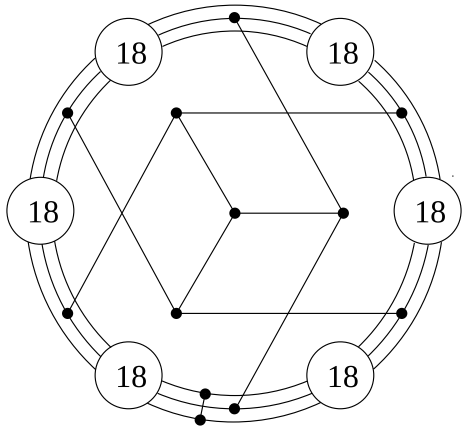








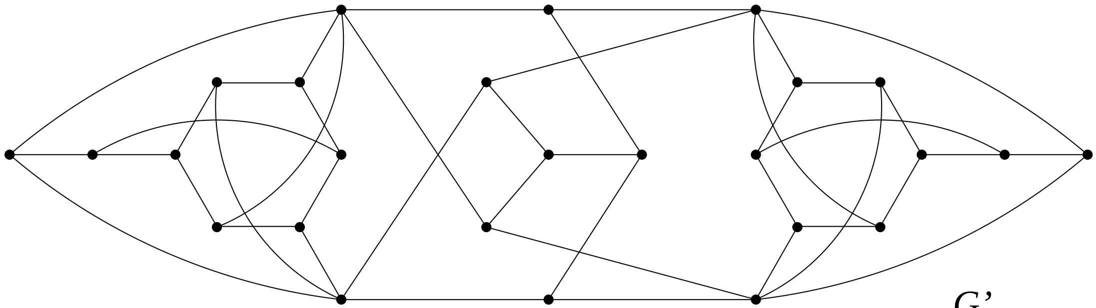
G_{118}



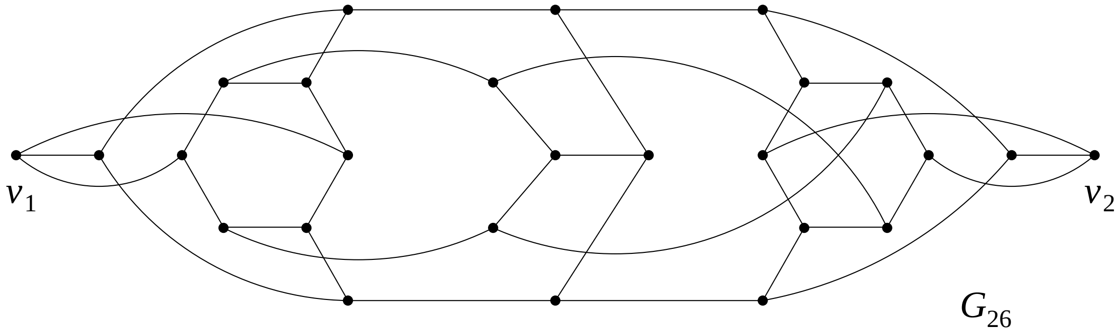
G_{120}

Theorem (K. 1996): *There exists a cyclically* 6*-edge-connected snark of any even order* ≥ 118*.*

Theorem (K. 2005): *It is an NP-complete problem to decide whether a cyclically* 6*-edge-connected cubic graph is* 3*-edge-colorable.*



G'_{26}



G_{26}

Snarks with large girths

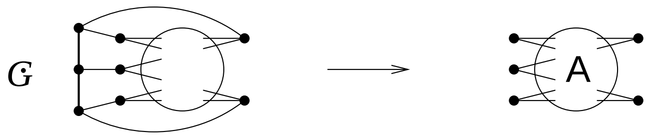
girth = length of the shortest circuit in a graph

Conjecture (Jaeger, Swart 1980): $\forall$ snark has girth ≤ 6.

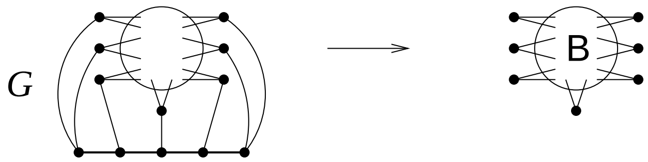
Theorem (Celmins 1984): *The smallest counterexample to the 5-flow conjecture is a snark with girth* ≥ 7.

Theorem (Goddyn 1985): *The smallest counterexample to the CDC conjecture is a snark with girth* ≥ 8.

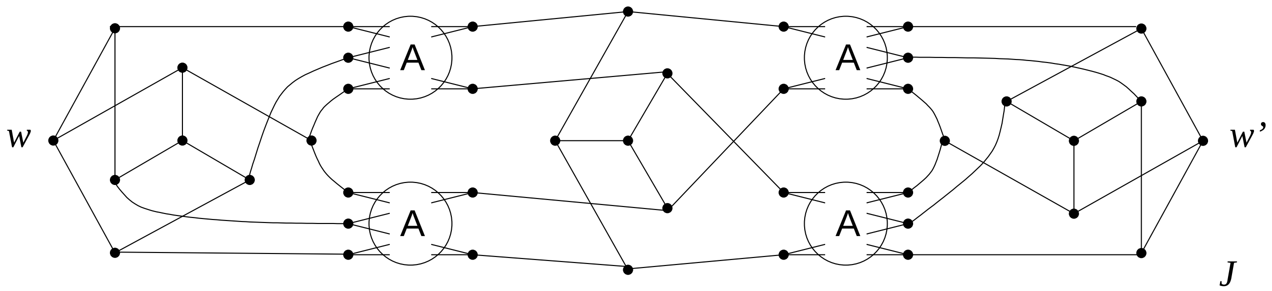
Theorem (K. 1996): *For any* $c \geq 7$*, there exists an infinite family of cyclically 5-edge-connected snarks of girth* c.

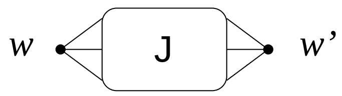


a symbolic representation of *A*

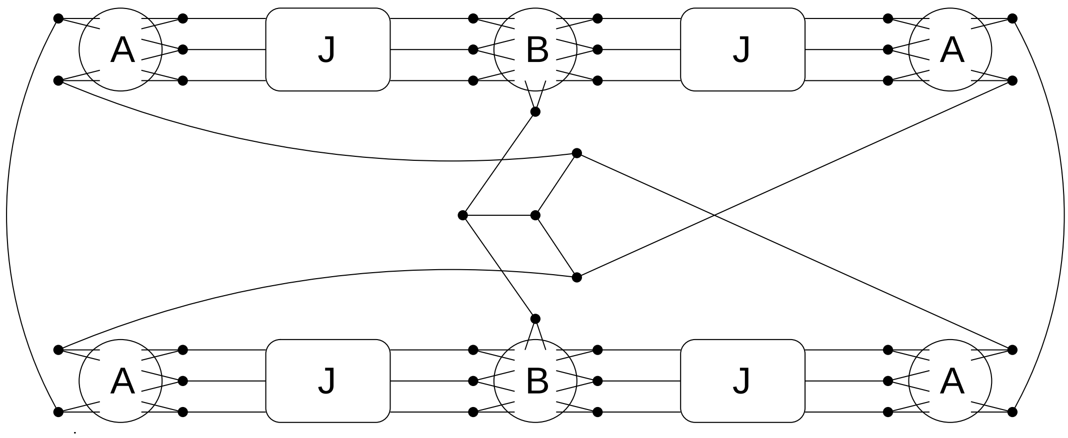


a symbolic representation of *B*





a symbolic representation of *J*

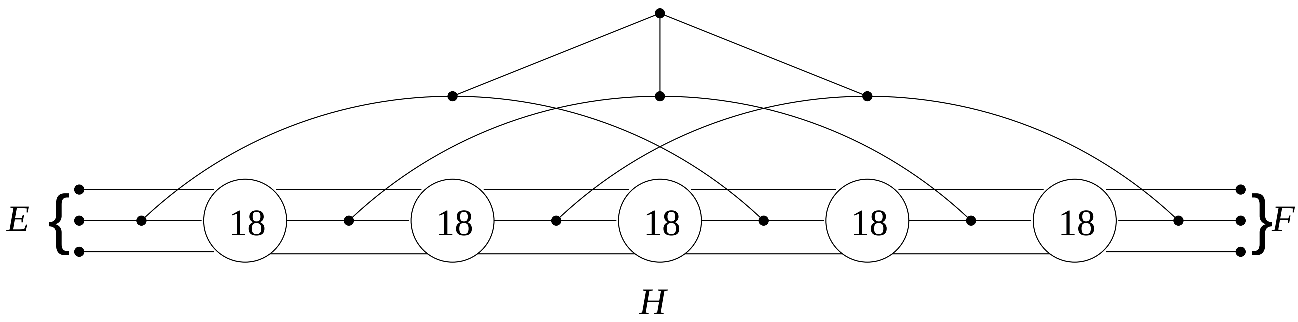


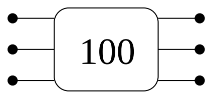
Conjecture (Jaeger, Swart 1980): Every cyclically 7-edge-connected cubic graph is 3-edge-colorable.

Theorem (K. 2004, 2005): *The smallest counterexample to the* 5*-flow conjecture must be a cyclically* 6-edge-connected *snark with girth at least* 9.

Theorem (Huck 2000): *The smallest counterexample to the CDC conjecture must be a snark with girth at least* 12.

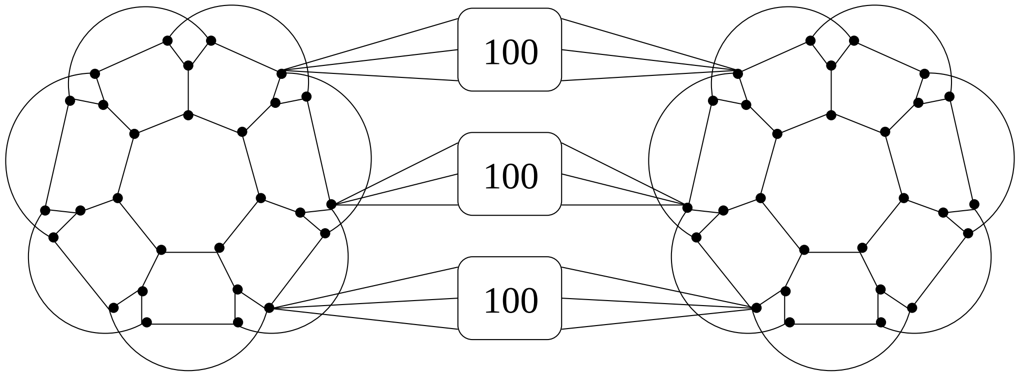
"Gluing" snarks





a symbolic representation of H

- H is 3-colorable;
- by any 3-edge-coloring, edges from E (F) have colors $1, 2, 3$;
- any two surjective mappings $E \to \{1, 2, 3\}$ and $F \to \{1, 2, 3\}$ can be extended to a 3-edge-coloring of H.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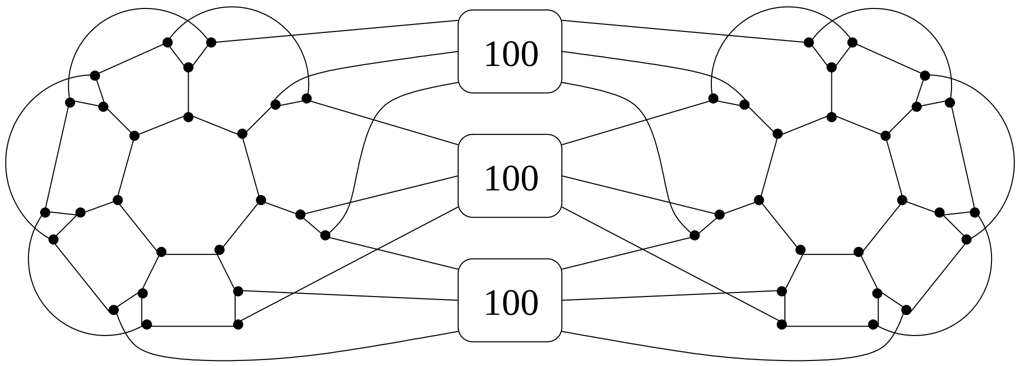
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Srongly uncolorable snarks

G = cubic graph

$\rho(G) = \min\{|U| : U \subseteq V(G),\ G - U$ is 3-edge colorable$\}$

Theorem (Vizing 1964): G is 4-edge-colorable.

$\sigma(G)$ = minimum k such that G has a 4-edge coloring with k edges colored by the fourth color.

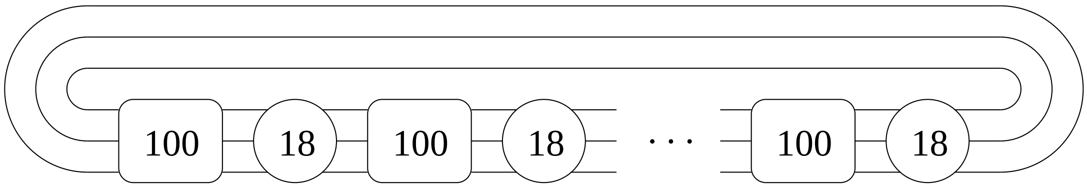
$\omega(G)$ = minimum k such that G can be covered by vertex-disjoint circuits so that among them are k of odd order.

Theorem (Huck, K. 1995): *If* $\omega(G) = 2$*, then* G *has a cycle double covering.*

Theorem (Huck 2001, Häggkvist, McGuinness 2005): *If* $\omega(G) = 4$*, then* G *has a cycle double covering.*

Theorem (Jaeger 1988): *If* $\omega(G) = 2$*, then* G *has nowhere-zero* 5*-flow.*

Theorem (K. 2002): *For every integer* $r > 0$*, there exists a cyclically* 6*-edge-connected snark of order* $118r$ *satisfying* $\rho(G), \sigma(G), \omega(G) \geq r$.



Vizing's theorem indicates a polynomial algorithm for 4-edge-coloring of a cubic graph G.

Doest there exists a polynomial algorithm giving a 4-edge-coloring of G so that the number of edges colored by the fourth color is at most $\sigma(G) + O(n^{1-\epsilon})$? No if $P \neq NP$.

Theorem (K. 2005): *It is an NP-complete problem to decide whether* $\rho(G), \sigma(G), \omega(G) \in [0, n^{1-\epsilon}]$.

$\rho_5(G) = \min\{|U| : U \subseteq V(G),\ G - U$ has a n.-z. 5-flow$\}$

Theorem (K. 1998, 2005): *If there exists a bridgeless graph without a nowhere-zero 5-flow, then:*

(1) *the problem to decide whether a graph has a nowhere-zero 5-flow is NP-complete;*

(2) $\forall\ r > 0$ *there exists a bridgeless graph* G_r *such that* $\rho_5(G_r) \geq r$*;*

(3) *the problem to decide whether* $\rho_5(G) \in [0, n^{1-\epsilon}]$ *is NP-complete.*

3-flows

Theorem (Grötzsch 1959): *Every planar graph without triangles is 3-colorable.*

3-Flow Conjecture (Tutte 1972): Every graph without 1- and 3-edge-cuts has a nowhere-zero 3-flow. $\Longleftrightarrow$ Every 4-edge-connected graph has a nowhere-zero 3-flow.

Theorem (Jaeger 1976): *Every 4-edge-connected graph has a nowhere-zero 4-flow.*

Weak 3-Flow Conjecture (Jaeger 1988): There exists $k \geq 4$ such that every k-edge-connected graph has a nowhere-zero 3-flow.

$\rho_3(G) = \min\{|U| : U \subseteq V(G),\ G - U$ has a n.-z. 3-flow$\}$

Theorem (K. 1998, 2005): *If there exists a k-edge-connected graph without a nowhere-zero 3-flow, then:*

(1) *the problem to decide whether a k-edge-connected graph has a nowhere-zero 3-flow is NP-complete;*

(2) $\forall\ r > 0$ *there exists a k-edge-connected graph G_r such that $\rho_3(G_r) \geq r$;*

(3) *the problem to decide whether $\rho_3(G) \in [0, n^{1-\epsilon}]$ is NP-complete for k-edge-connected graphs.*

for $k = 3$, the statement holds for planar graphs

Theorem (K. 2002): *The following statements are equivalent*

(1) *Every* 4*-edge-connected graph has a nowhere-zero* 3*-flow.*

(2) *Every bridgeless graph with at most three edge cuts of cardinality* 3 *has a nowhere-zero* 3*-flow.*

(3) *Every bridgeless graph* G *with vertices* v_1, v_2, v_3 *such that there is no* 3*-edge-cut* C *of* G *where* $G - C$ *has a component containing all* v_1, v_2, v_3 *has a nowhere-zero* 3*-flow.*

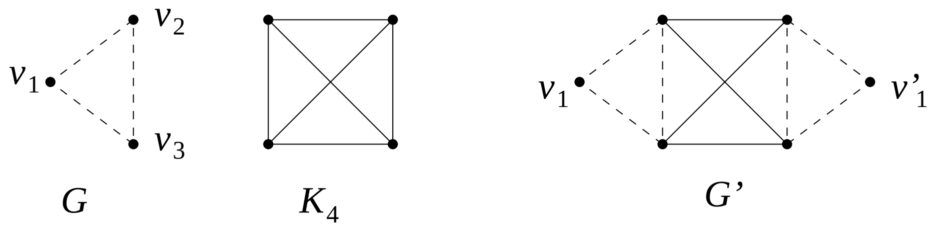
(1) holds for planar graphs (Grötzsch 1959)

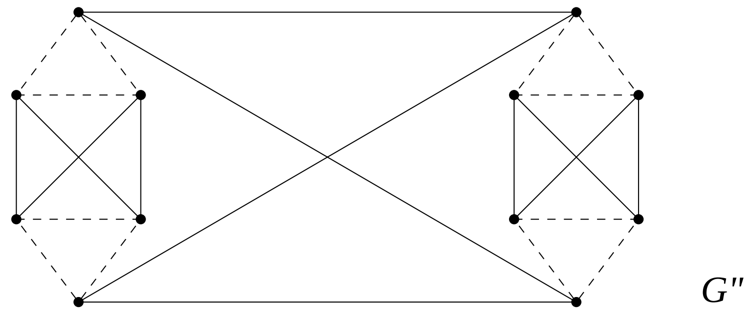
(2) holds for planar graphs (Grünbaum 1963, Aksionov 1974, Borodin 1997)

Proof: $(3) \implies (2) \implies (1)$ - trivial

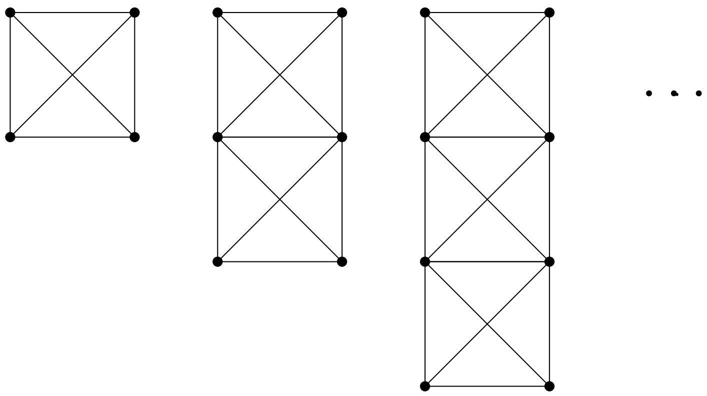
$(1) \implies (3)$ - nontrivial

G - a counterexample to $(3) \implies G''$ - a counterexample to (1)





Theorem (K. 2002): There exists and infinite family of planar graphs with exactly four 3-edge-cuts not admitting nowhere-zero 3-flows.



Theorem (K. 2001): *The following statements are equivalent*

(1) *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

(2) *Every 5-edge-connected graph has a nowhere-zero 3-flow.*

A graph G is called A**-connected** if $\forall$ orientation of G and $\forall\ b : V(G) \to A$ such that $\sum_{v \in V(G)} b(v) = 0$

$\exists\ \varphi : E(G) \to A - \{0\}$ such that $\forall\ v \in V(G)$

$$\sum_{e \text{ enters } v} \varphi(e) - \sum_{e \text{ leaves } v} \varphi(e) = b(v).$$

If G is A-connected $\Longrightarrow$ G has a nowhere-zero A-flow

Conjecture (Jaeger, Linial, Payan, Tarsi 1992):

Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.

Circular flow numbers

Graph G has a **circular flow number** r if r is the smallest real such that the edges of G can be oriented and assigned real numbers from $[1, r-1]$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph.

Conjecture (Mohar): Every snark different from Petersen graph has circular flow number < 5.

Theorem (Máčajová, Raspaud 2005): *There are infinitely many snarks with circular flow number* $= 5$.

Circuits in graphs

A **cycle double covering** (**CDC**) of a graph G is a family of circuits $\mathcal{L} = \{C_1, \ldots, C_n\}$ in G such that each edge of G is contained in exactly two circuits from $\mathcal{L}$.

CDC Conjecture (Seymour 1978, Szekeres 1973): Every bridgeless graph has a CDC.

If a cubic graph G has a 3-edge-coloring $\Longrightarrow$ *G has a* CDC.

A circuit C in a graph G is called **dominating** if each edge of G is incident with a vertex from C.

(1) Conjecture (Fleischner 1984): Every cyclically 4-edge-connected cubic graph has – either a dominating circuit,

– or a 3-edge-coloring.

(2) Conjecture (Sabidussi 1985): Given an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition $\mathcal{S}$ of G into circuits so that consecutive edges in T belongs to different circuits in $\mathcal{S}$.

(3) Conjecture (Fleischner 1984): If C is a dominating circuit in a cyclically 4-edge-connected cubic graph G, then there exists a cycle double cover of G which includes C.

(2) $\Longleftrightarrow$ (3) – Fleischner 1984

(1) & (3) $\Longrightarrow$ CDC conjecture – Jaeger 1985

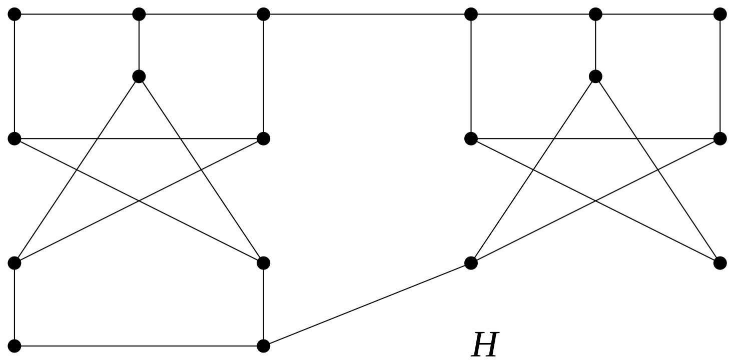
Theorem (Seymour 1979): *If C is a circuit in a 3-edge-colorable cubic graph G, then G has a* CDC *which includes C.*

A circuit C in a graph G is **stable** if there does not exist another circuit D so that $V(C) \subseteq V(D)$.

If $\nexists$ snark with a stable circuit $\Longrightarrow$ CDC conjecture

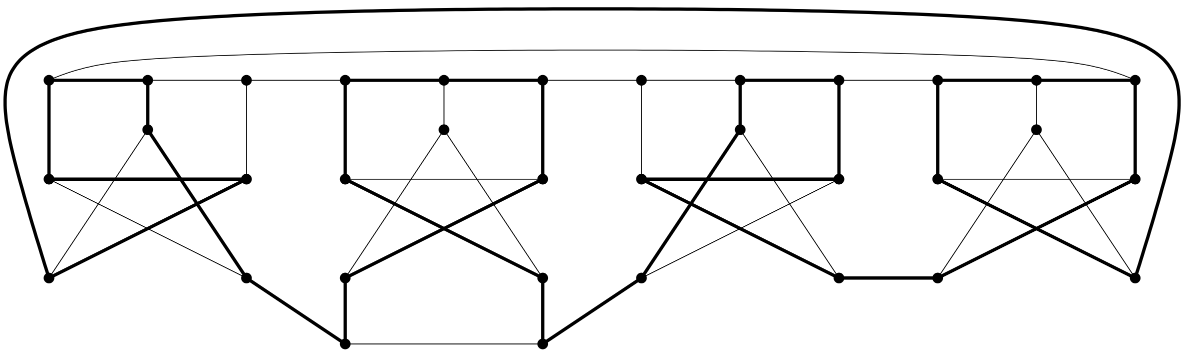
If $\nexists$ snark with a stable dominating circuit $\Longrightarrow$ (2), (3)

Lemma (K. 2000): *If a cubic graph G contains the graph H as an induced subgraph, then G is not 3-edge-colorable.*



H

Theorem (K. 2001): *For any nonnegative integers k, m there exists a snark of order $34+8k+18m$ having a stable dominating circuit of length $30+7k+16m$.*



(1) Conjecture (Fleischner 1984): Every cyclically 4-edge-connected cubic graph has – either a dominating circuit,

– or a 3-edge-coloring.

(4) Conjecture (Ash, Jackson 1984): Every cyclically 4-edge-connected cubic graph has a dominating circuit.

(5) Conjecture (Thomassen 1985): Every 4-connected line graph is hamiltonian.

(6) Conjecture (Matthews, Sumner 1984): Every 4-connected claw-free graph is hamiltonian.

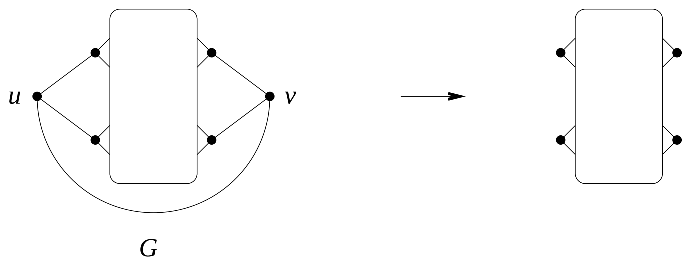
(4) $\iff$ (5) – Fleischner, Jackson 1989

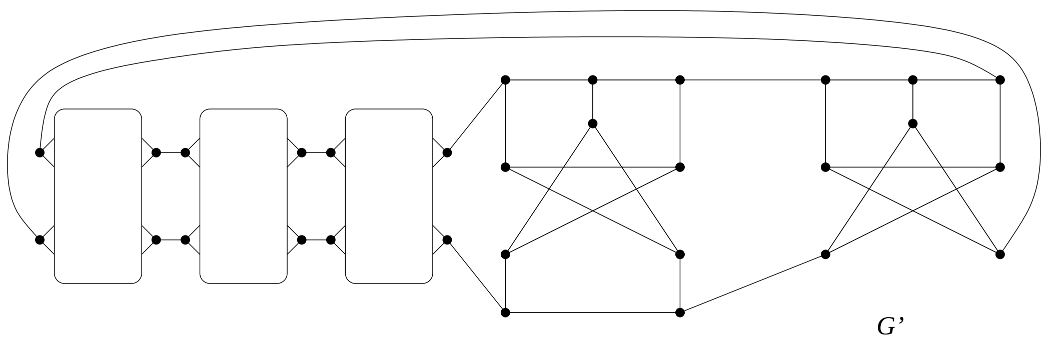
(5) $\iff$ (6) – Ryjáček 1997

(1) $\iff$ (4) – K. 2000

(4) an (5) hold for planar graphs (Tutte 1956)

Theorem (K. 2000): *If there exists a 4-edge-connected cubic graph G with no dominating circuit $\Longrightarrow$ there exists a 4-edge-connected cubic graph G' without an edge-3-coloring and with no dominating circuit.*





Theorem (Fleischner, K. 2002, Kužel, Xiong): *Every cyclically* 4*-edge-connected cubic graph has a dominating circuit* $\Longleftrightarrow$ *any two edges in a* 4*-edge-connected cubic graph are contained in a dominating circuit.*

Theorem (Kužel, Xiong 2005): *Every* 4*-connected line graph is hamiltonian* $\Longleftrightarrow$ *every* 4*-connected line graph is hamiltonian connected.*

Theorem(K. 2002): *The following statements are equivalent.*

(a) *Every 4-connected claw-free graph is hamiltonian.*

(b) *Vertices of every 4-connected claw-free graph of order n can be covered by $o(n)$ vertex-disjoint paths.*

(c) *Every 4-connected line graph is hamiltonian.*

(d) *Vertices of every 4-connected line graph of order n can be covered by $o(n)$ vertex-disjoint paths.*

(e) *Every cyclically 4-edge-connected cubic graph has a dominating circuit.*

(f) *Every cyclically 4-edge-connected cubic graph of order $2n$ has a dominating subgraph consisting of $o(n)$ paths.*

(g) *Every cyclically 4-edge-connected non 3-edge-colorable cubic graph has a dominating circuit.*

(e) *Every cyclically 4-edge-connected non 3-edge-colorable cubic graph has a dominating subgraph consisting of $o(n)$ paths.*

Conjecture (Barnette 1969): Every 3-connected cubic planar graph is hamiltonian.

Theorem (Kelmans 1986, K. 2002): *The following statements are equivalent.*

(a) *Every 3-connected cubic planar graph is hamiltonian.*

(b) *Every cylically 4-edge-connected cubic planar graph is hamiltonian.*

(c) *Any two edges in a cylically 4-edge-connected cubic planar graph are contained in a hamiltonian circuit.*

(d) *Vertices of every cylically 4-edge-connected cubic planar graph of order $2n$ can be covered by $o(n)$ vertex-disjoint paths.*

Conjecture (Jackson 1993): K_5 is the only 4-connected eulerian graph with an even number of edges but no even circuit decomposition.

Theorem (Rizzi 2001): *There exists an infinite family of* 4*-connected eulerian graphs with an even number of edges but no even circuit decomposition.*

Some open problems

Every bridgeless graph has a nowhere-zero 5-flow.

Every 5-edge-connected graph has a nowhere-zero 3-flow.

Every bridgeless graph has a CDC.

Every 4-connected line graph is hamiltonian.

For an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition $\mathcal{S}$ of G into circuits so that consecutive edges in T belongs to different circuits in $\mathcal{S}$.