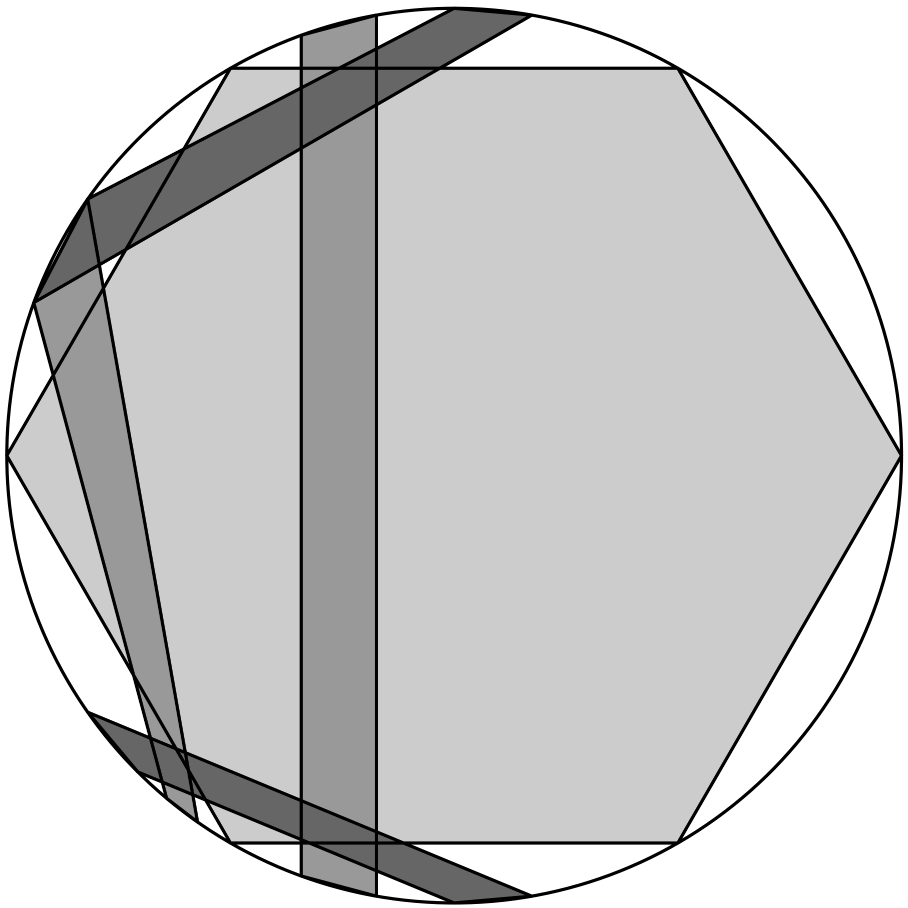
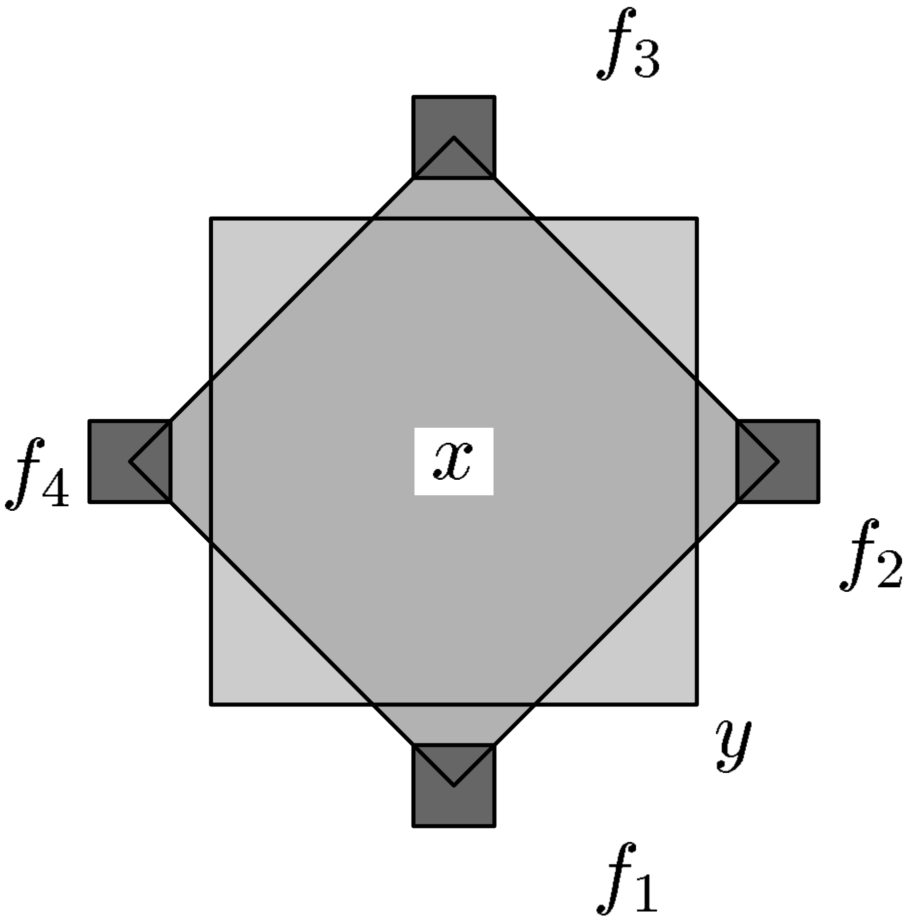
Complexity and complicacy questions of graphs representable by polygons

(with J. Kratochvíl)

Intersection graphs - we take sets of points
for each set add a vertex,
two vertices are adjacent if the corresponding two sets have a nonempty intersection.



Polygon-circle (PC) graphs are intersection graphs of convex polygons inscribed to a circle. CONV-graphs are intersection graphs of convex polygons in the plane.



Def.: For class of PC-graphs:
- complicacy of a graph G - minimum k such that G is the intersection graph of convex k-gons inscribed to a circle
- denote it $cmp(G)$;

$cmp(n)$ is maximum complicacy over PC-graphs on n vertices.

For class of CONV-graphs we define complicacy analogically.

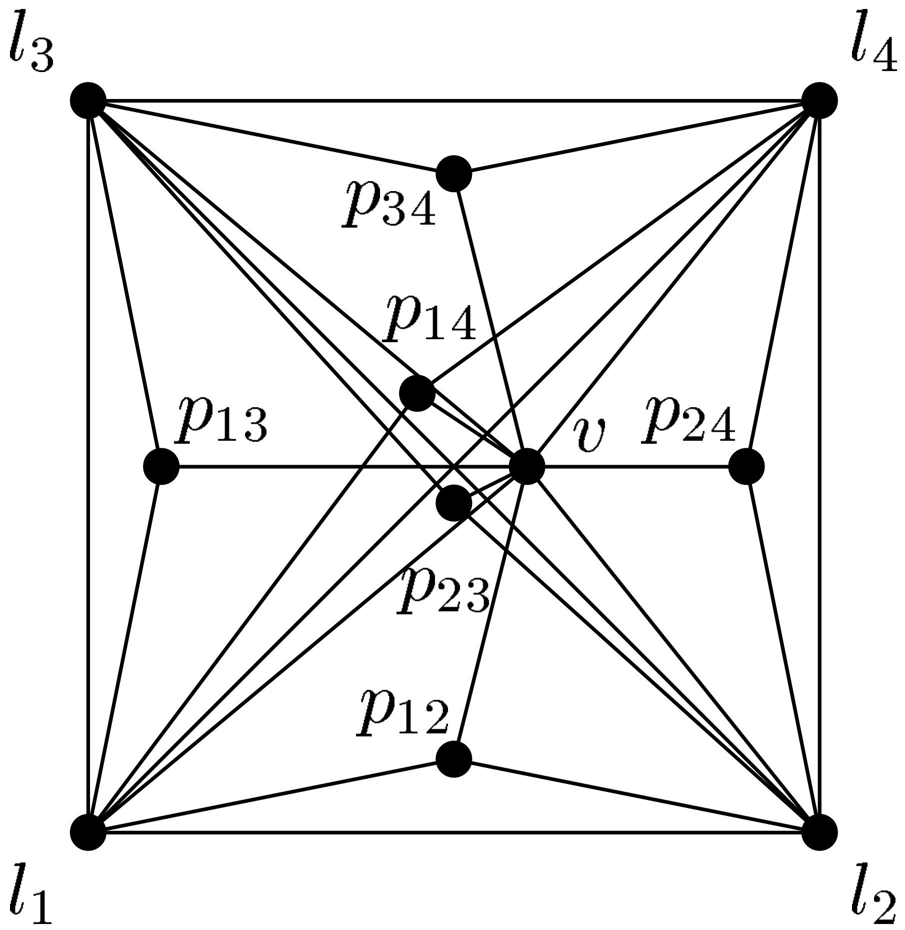
Notation:
$x\ldots$ vertex,
P_x ... polygon representing x.

Theorem: For PC-graphs it holds:
$cmp(n) = n - \log n + o(\log n)$.

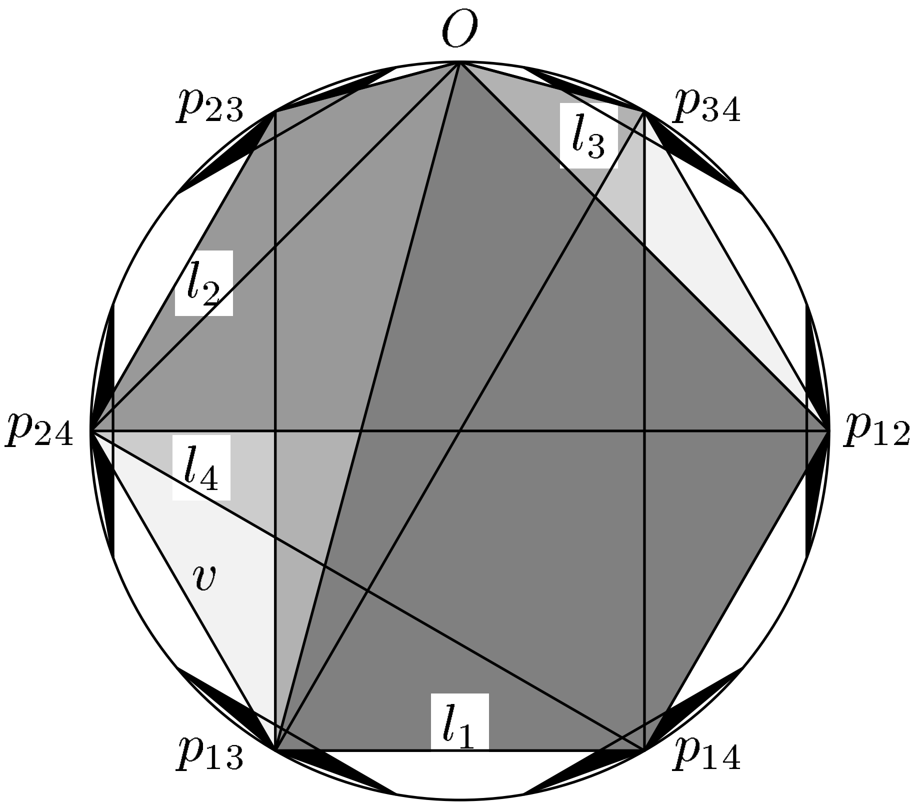
Lower bound: By explicit construction, $V(G) = P \cup L \cup \{v\}$:

- L – set of even number of vertices forming a clique.
- P – independent set indexed by $\frac{|L|}{2}$-tuples from L.

  $\forall_{p \in P, l \in L}$ pl forms an edge iff l is in a tuple indexing p.
- v – a vertex adjacent to all elements of P and L.



Claim: Graph G from the above construction has $\mathrm{cmp}(G) = |P|$.



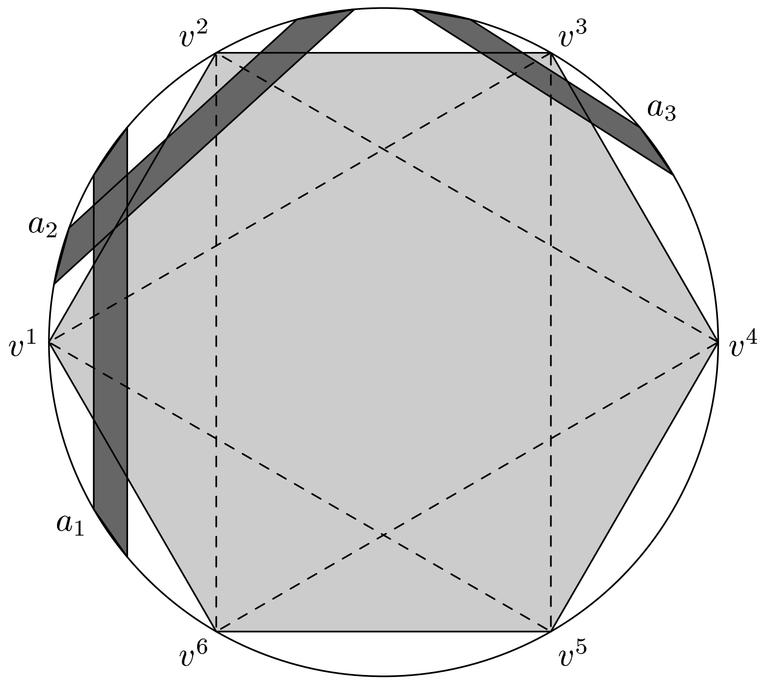
O
p_{23}
p_{34}
l_3
l_2
p_{24}
p_{12}
l_4
v
l_1
p_{13}
p_{14}

Upper bound: Easy upper bound – $\Delta(G)$ holds even for CONV-graphs.

Def.: Given a polygon P with vertices $P_1, ...P_x$ each segment (of the geometrical circle) between P_i and P_{i+1} (resp. P_1, P_x) is called *P-segment*.

Prop.: For class of PC-graphs $cmp(n) \leq n - \log(n) - o(\log(n))$.

For a fixed PC-graph G on n vertices we show, that cmp$(G) \leq n - \log n - o(\log(n))$. Consider minimal PC-representation of G. Consider fixed vertex $v \in V(G)$. For P_v: $A' = \{u \in V(G) | P_u$ lies in at most two consecutive P_v-segments$\}$.

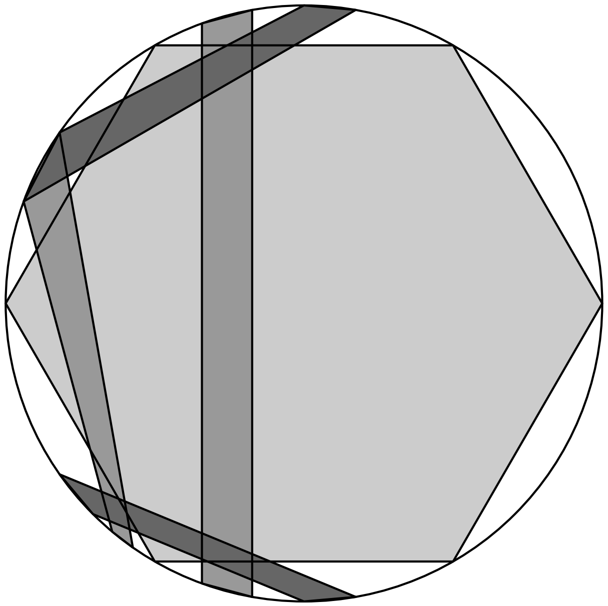


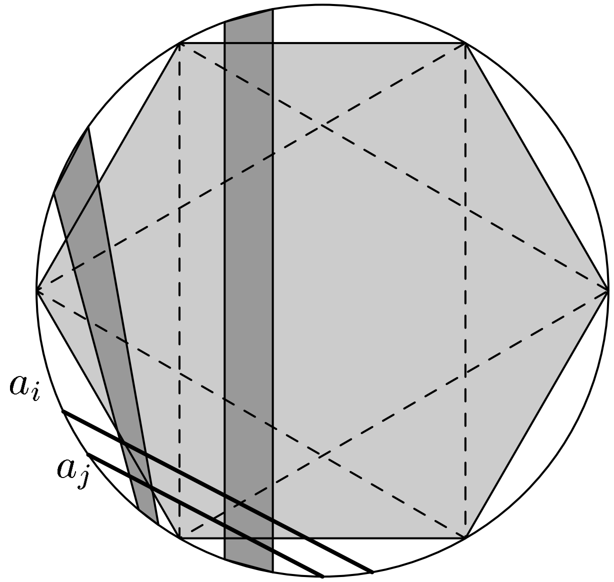
Assigning label (natural n.) to each corner of P_v for a fixed representation we define index of vertex in A' – denote by $i(u)$. Consider representation with previous numbers of corners with

$$\min \sum_{u \in A} i(u).$$

We may move the polygons.

Now consider $A \subseteq A'$ largest subset of vertices such that no two of them have the same index.





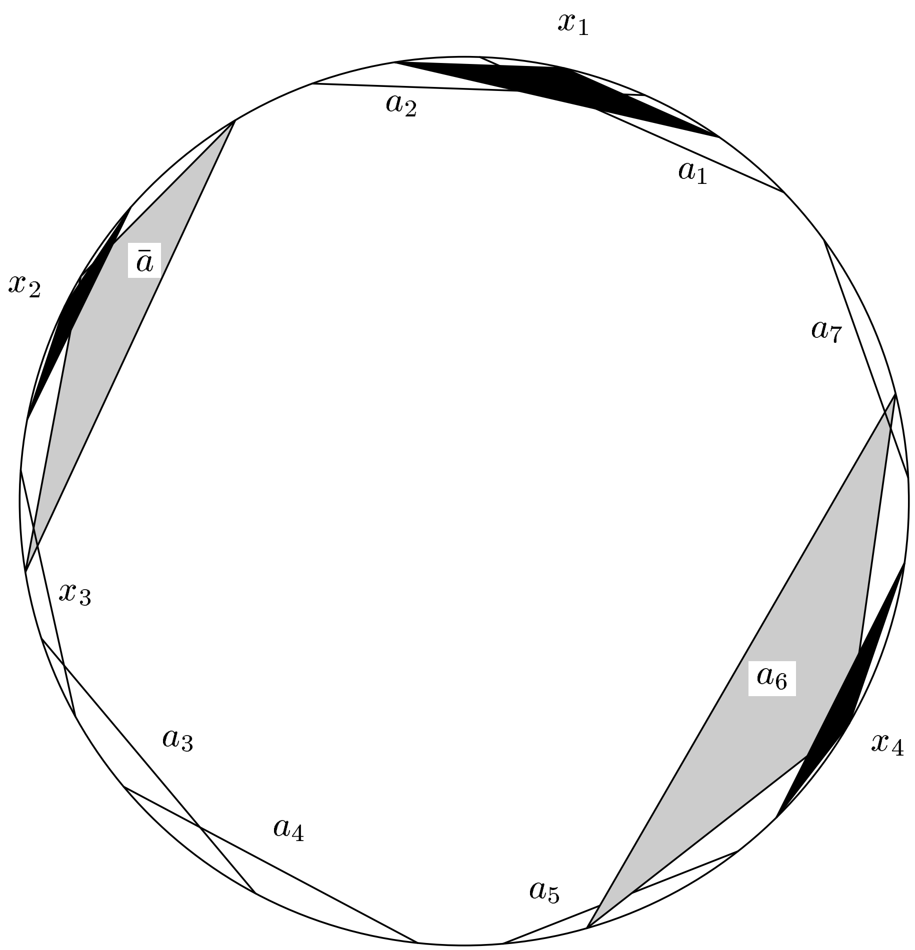
Define $A_1 = \{u \in A | \forall_{x \in A} ux \notin E(G)\}$ and $A_2 = A \setminus A_1$.

Now: $|A_1| \leq 2^{V(G) \setminus A}$ (different sets of neighbours, otherwise we move the polygons).

For A_2:
We can intersect two polygons from A_2 at once (impossible)
or under one of them there is another polygon (not in A),
or there is a segment under "both of them" we must not intersect.

$\{a_1, a_2, ...\} = A_2$ and we can divide A_2 to at most triples such that each has some "witness" outside A_2, thus:

$$3 \cdot |A_2| \leq |V(G) \setminus A|.$$



Complicacy forced by A_1 grows much faster than cmp forced by A_2.

CONV-graphs:

Claim:
For CONV-graphs: $n-1 \geq cmp(n) \geq \frac{n-10}{6}$.

Upper bound: Similar to the simple upper bound for PC-graphs, while the more sophisticated one fails.

Lower bound: Partially similar to the simple lower bound for PC-graphs (explicit construction - infinite sequence with proper cmp).

Recognition problem for PC and CONV-graphs:

PC-graphs with complicacy 2 – polynomial (Spinrad, Bouchet),

PC-graphs with complicacy $k > 2$ – NP-complete (Kratochvíl, MP)

PC-graphs – ? (Koebe announced polynomial algorithm).

CONV-graphs with complicacy 2 (SEG) – NP-hard,

CONV-graphs – NP-hard,

CONV-graphs with limited complicacy – NP-hard.

Open problems:

Tight (or better) bounds of complicacy of CONV-graphs, recognition of PC-graphs.