Magic labelings
of
regular graphs

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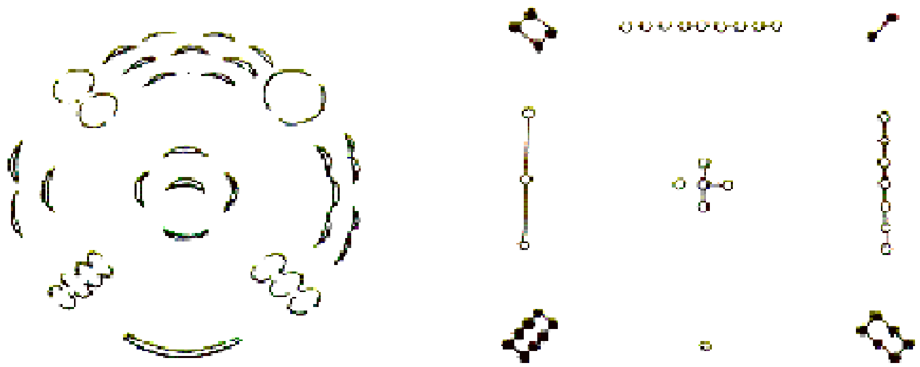
Overview

- Part 1 : Introduction
  Magic squares and magic graph labelings, definitions and examples
- Part 2: Known results
  Classes of magic graphs, copies and products of regular graphs
- Part 3: Results for VAMT labelings
  Theorem, exmaple
- Part 4: Results for VMT labelings
  Theorem, exmaple, VMT from VAMT
- Part 5: Open problems
  Approaching MacDougall’s conjecture

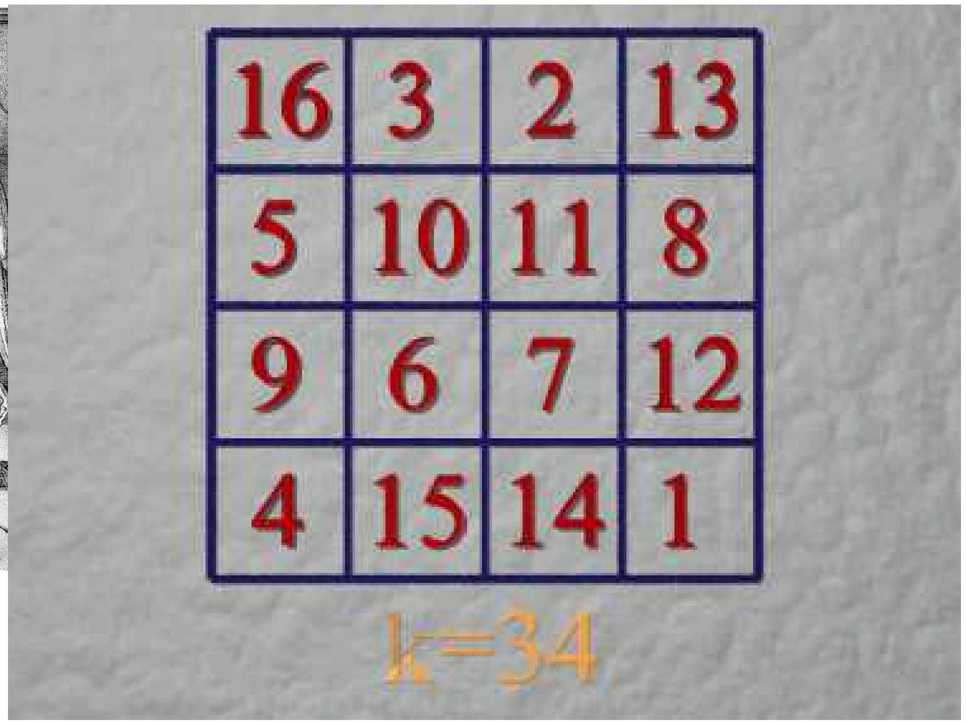
Introduction

- Notes on history, magic squares
- Magic graph labelings
- Definitions of VMT and VAMT labelings + examples

Magic squares (1/5)



"Lo Shu" - China, book *Yih King*, 2200 BC

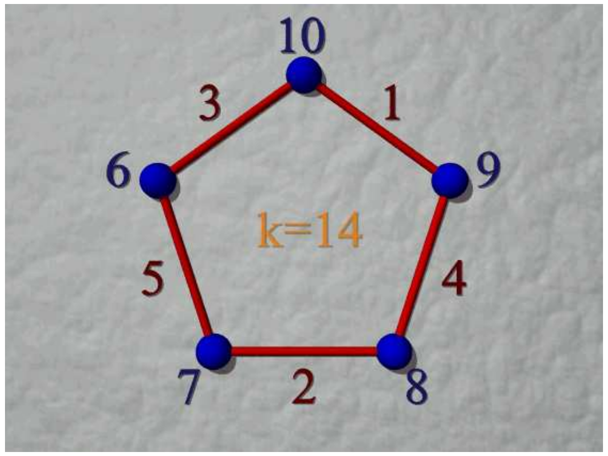


Middle age, Albrecht Dürer *Melancholia*, 1514

Various magic-type labelings (2/5)

Which labels do we sum?

- vertex magic
- edge magic



What do we label?

- edge labeling
- vertex labeling
- total labeling

Wallis, Magic Graphs (2001)

We focus on three magic type labelings

- vertex magic total labeling (VMT)
- vertex antimagic total labeling (VAMT)

Definition

Let $G(V, E)$ be a graph. A one-to-one mapping

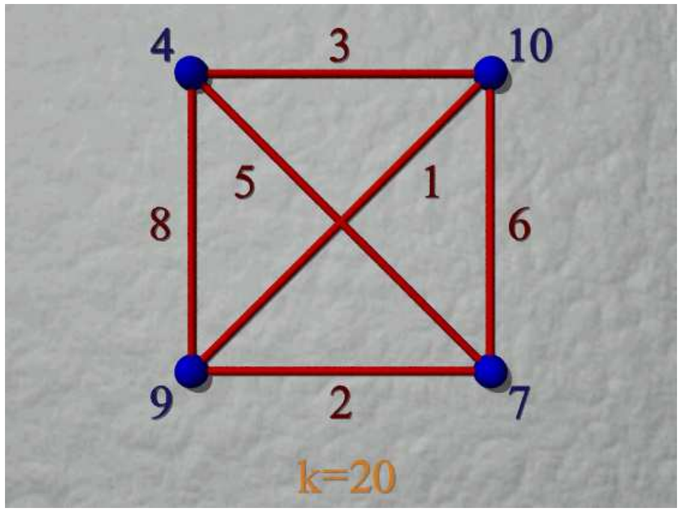
$\lambda : V \cup E \to \{1, 2, \ldots, |V|+|E|\}$

is a *vertex magic total labeling* (VMT labeling) if there is a constant k so that for every vertex x is

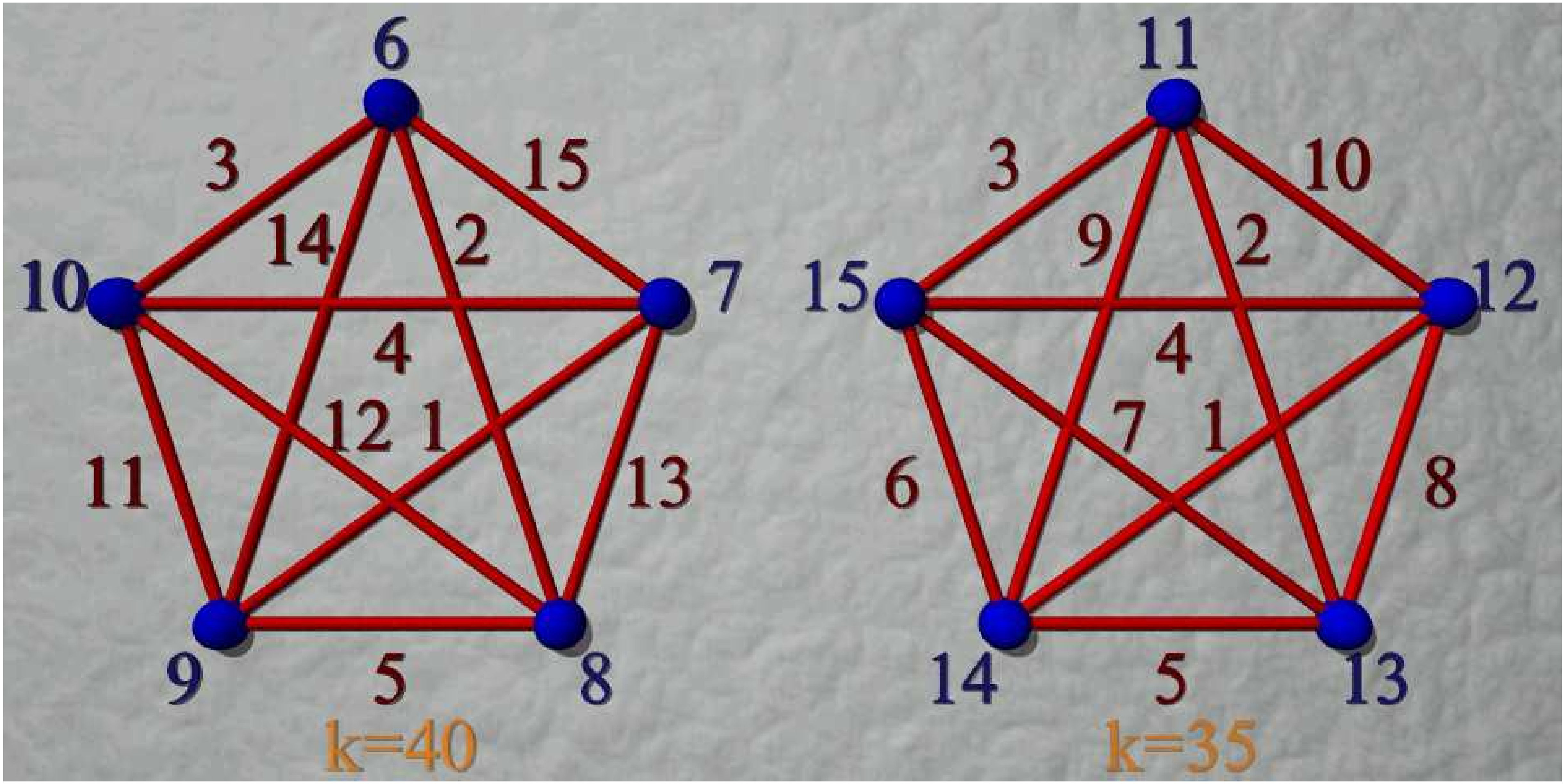
$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k$$

where k is the *magic constant* for λ.

The sum associated to a vertex x is called the *weight of* x and denoted by $w_\lambda(x)$.



Vertex magic total labeling of K_4.



VMT labelings of K_5 with different magic constants

Definition

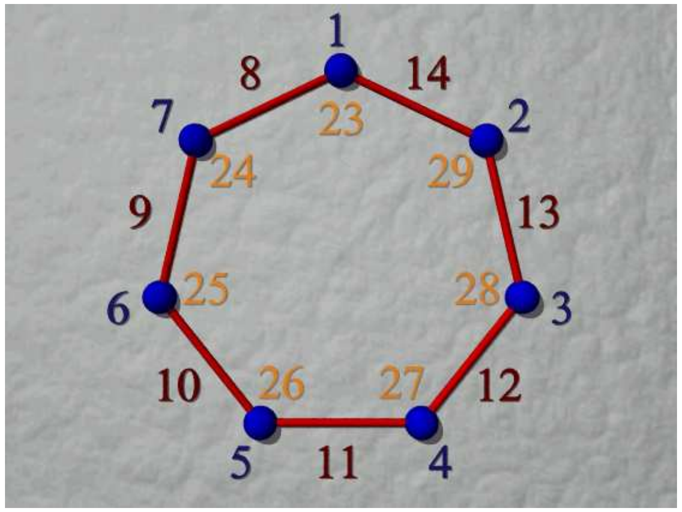
Let $G(V, E)$ be a graph. A one-to-one mapping

$$\lambda : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$$

is a *vertex antimagic total labeling* if if the sums

$$w_\lambda(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy)$$

are distinct for all vertices of G.



Vertex magic total labeling of C_7.

The labeling λ is called an (s, d)-*vertex antimagic total labeling (VAMT) of* G if the weights $w_\lambda(x)$ form an arithmetic progression

$$s, s + d, \ldots, s + (v - 1)d.$$

Known results

- VMT and VAMT classes of graphs
- Copies and products of graphs
- Theoretical bounds
- MacDougall’s Conjecture

Table of known VMT labelings

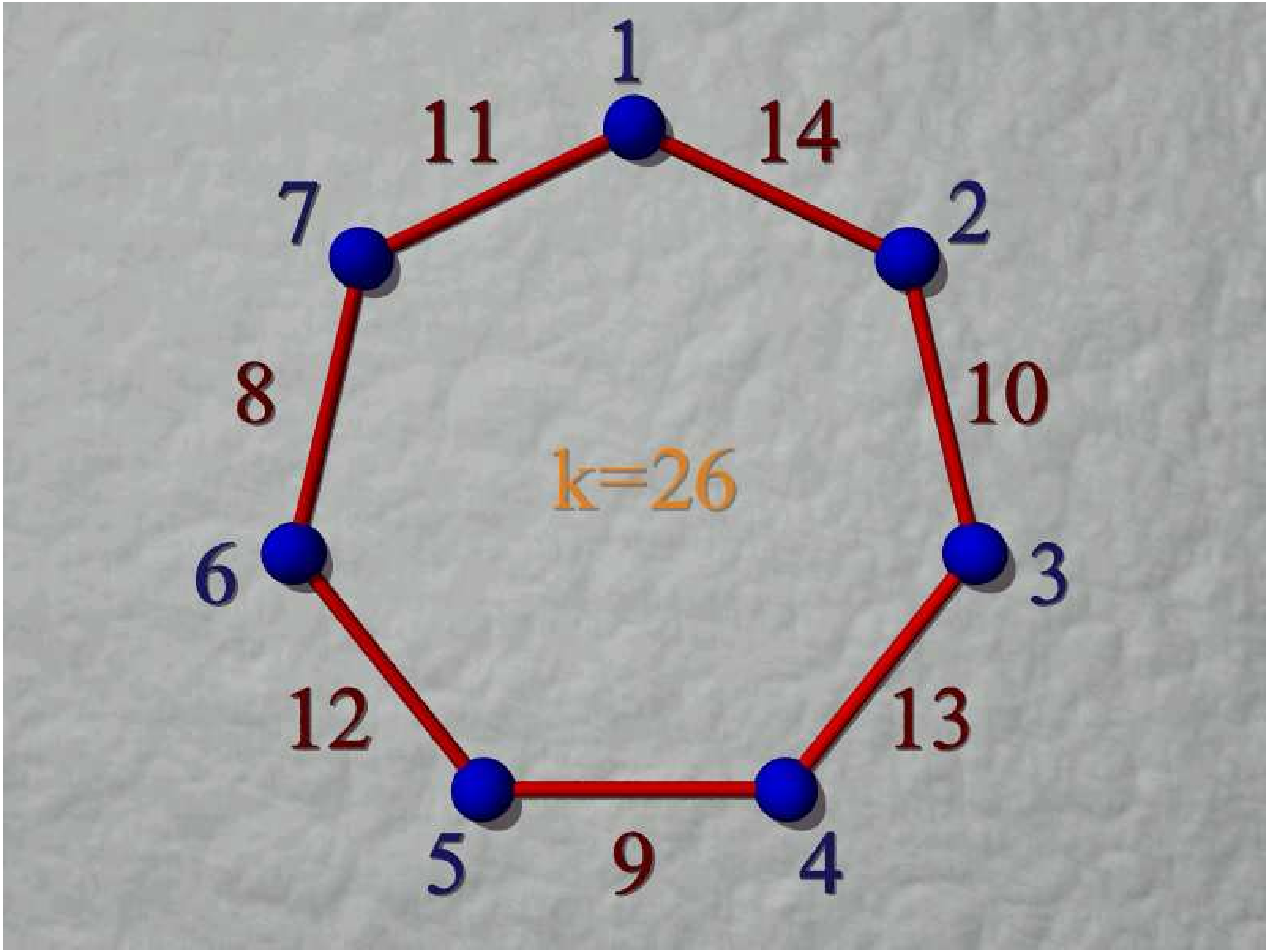
Joseph A. Gallian, A Dynamic survey of graph labeling, The Electronic Journal of Combinatorics, DS 6 (2003).

Graph	*Labeling*	*Notes*
C_n	VMT	[DMSW]
P_n	VMT	$n > 2$ [DMSW]
$K_{m,m}$	VMT	$m > 1$ [DMSW][LM]
$K_{m,m} - e$	VMT	$m > 2$ [DMSW]
$K_{m,n}$	VMT	iff $\|m - n\| \leq 1$ [DMSW]
	not VMT	if $n > m + 1$ [MMSW]
K_n	VMT	n odd [DMSW]
		$n \equiv 0 \pmod 4$, $n > 2$ [LM]
		$n \equiv 2 \pmod 4$, $n > 2$
Petersen $P(n, k)$	VMT	for all n and k [BMS]
prisms $C_n \times P_2$	VMT	for all n [SlM]
W_n	VMT	iff $n \leq 11$ [DMSW]
F_n	VMT	iff $n \leq 10$ [DMSW]

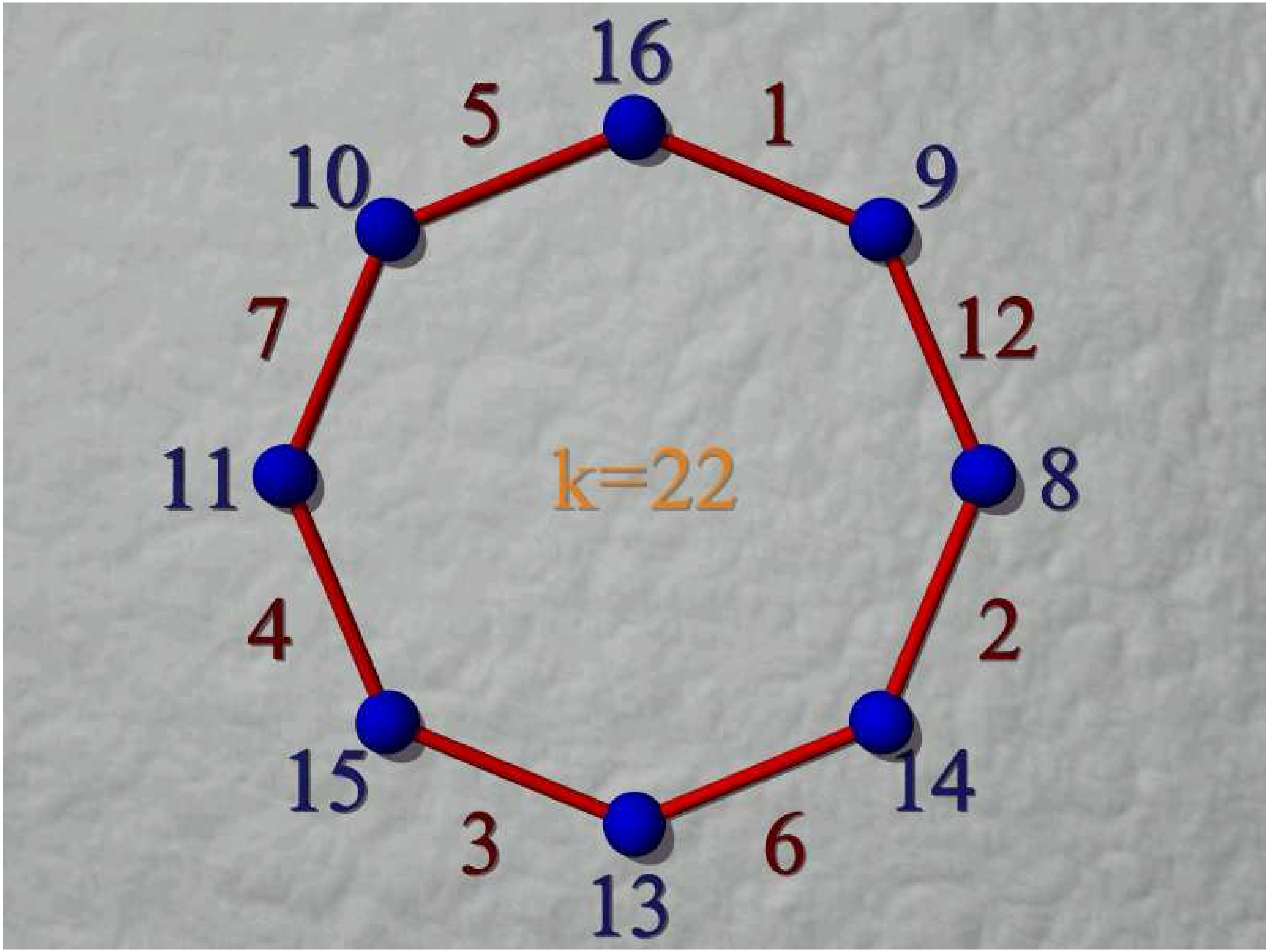
Table continued

Graph	*Labeling*	*Notes*
friendship graphs	VMT	iff # of triangles ≤ 3 [DMSW]
$G + H$	VMT	$\|V(G)\| = \|V(H)\|$ and $G \cup H$ is VMT [Wal1]
unions of stars	VMT	[Wal1]
nG	VMT	n odd, G regular of even degree, VMT [Wal2]
nG	VMT	G is regular of odd degree, VMT [Wal2]
$C_n \times C_{2m+1}$	VMT	[FKK1]
$K_5 \times C_{2n+1}$	VMT	[FKK2]
$G \times C_{2n}$	VMT	G $2r+1$-regular VMT [Kov1]
$G \times K_5$	VMT	G $2r+1$-regular VMT [Kov1]
$G \times H$	VMT	G r-regular VMT H $2s$-regular SPM [Kov2]

Odd cycle



VMT labeling of C_7



VMT labeling of C_8

Table of known VAMT labelings

Joseph A. Gallian, A Dynamic survey of graph labeling, The Electronic Journal of Combinatorics, DS 6 (2003).

Graph	*Labeling*	*Notes*
C_n	(s,d)-VAMT	variety of s,d [BBDMSS]
P_n	(s,d)-VAMT	variety of s,d [BBDMSS]
$K_{n,n}$	$(s,1)$-VAMT	$n \geq 3$ [BBDMSS]
A class of quartic graphs R_n	$(s,1)$-VAMT	[BHL]
Petersen graph $P(n,k)$	(s,d)-VAMT	[MB]
prisms $C_n \times P_2$	(s,d)-VAMT	[BBDMSS] [BH]
antiprisms	(s,d)-VAMT	[BBMMSS2]
W_n	not (s,d)-VAMT	for $n > 20$ [LMSS]

Theorem by Wallis

Let G be an r-regular VMT graph other than K_1.

(1) If r is even then nG has a VMT labeling for n odd.

(2) If r is odd then nG has a VMT labeling for every n.

W. D. Wallis (2000)

Unified approach (also SPM)

Let r be an integer at least 3. Let G be an r-regular graph with a proper edge r coloring, which has an SPM labeling λ.

(1) If r is odd then nG has an SPM labeling for n odd.

(2) If r is even then nG has an SPM labeling for every n.

P. K. (2004)

Theorem

Let G be an r-regular VMT graph on m vertices. Let H be an s-regular class 1 SPM graph on n vertices. Suppose

(1) r is odd or n is odd,

(2) s is even or m is odd,

then there exists a VMT labeling for $G \times H$.

Theorem

Let G be an r-regular class 1 SPM graph on m vertices. Let H be an s-regular class 1 SPM graph on n vertices. Suppose

(1) r is even or n is odd,

(2) s is even or m is odd,

then there exists an SPM labeling of $G \times H$.

Theorem

Let r, s, p, q be positive integers, $p, q \geq 2$.
Let G be an rp-regular VMT graph on m vertices which can be factorized into r p-regular factors.
Let H be an sq-regular graph on n vertices which can be factorized into s q-regular factors and which has a VMT labeling with vertices labeled by consecutive integers. Suppose

(1) r is even or n is odd,

(2) s is even or m is odd,

then there exists a VMT labeling of $G \times H$.

Bounds on the paramters s and d (9/11)

Theorem

Let G be a graph with smallest vertex degree δ and largest vertex degree Δ. Let λ be an (s,d)-vertex antimagic total labeling of G. Then

(1) $d \leq \frac{(\Delta+1)(2(v+e)-\Delta)-(\delta+1)(\delta+2)}{2(v-1)}$

(2) $\frac{(\delta+1)(\delta+2)}{2} \leq s \leq \frac{1}{2}(\Delta+1)(2(v+e)-\Delta)-(v-1)d.$

Corollary

Let G be an r-regular graph. If there exists an (s,d)-vertex antimagic total labeling of G then $d \leq \frac{(r+1)(v+e-r-1)}{v-1}$ and $\frac{(r+1)(r+2)}{2} \leq s \leq \frac{1}{2}(r+1)(2(v+e)-r)-(v-1)d.$

M.Baca, F. Bertault, J.A. MacDougall, M.Miller, Rinovia, Simanjuntak, and Slamin (2000)

Bounds on the magic constant (10/11)

Corollary

Let G be a graph with smallest vertex degree δ and largest vertex degree Δ. Let λ be a vertex magic total labeling of G with the magic constant k. Then

$$\frac{(\delta+1)(\delta+2)}{2} \leq k \leq \frac{1}{2}(\Delta+1)(2(v+e)-\Delta).$$

Corollary

Let G be a regular graph with v vertices and e edges. Let λ be a vertex magic total labeling of G with the magic constant k. Then

$$\frac{\binom{v+e+1}{2}+\binom{e+1}{2}}{v} \leq k \leq \frac{2\binom{v+e+1}{2}-\binom{v+1}{2}}{v}.$$

J.A. MacDougall, M. Miller, Slamin, and W.D. Wallis. (2000)

MacDougall's conjecture (11/11)

Conjecture

Any regular graph other than K_2 or $2K_3$ is VMT.

D. McQuillan (2004)

Rephrased conjecture

Any r-regular graph for $r > 1$, with the exception of $2K_3$, has a VMT labeling.

P. K. (2004)

Results on VAMT labelings

- Theorem
- Example
- Further results

Even regular graphs are VAMT (1/8)

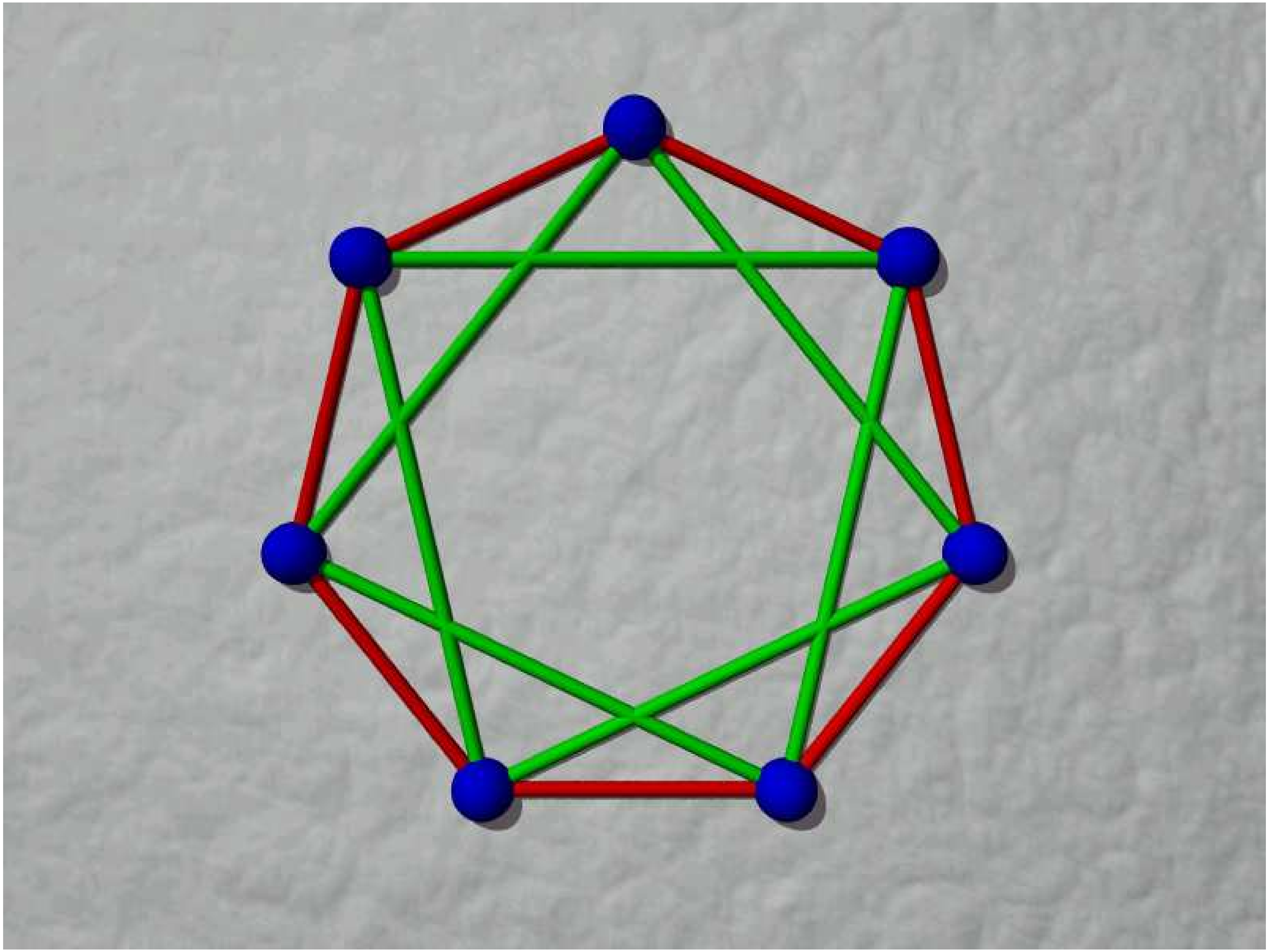
Theorem

Let G be a $2r$-regular graph with vertices $x_1, x_2, \ldots, x_n$. Then there exists an $(s,1)$-VAMT labeling λ of G such that

- $\lambda(x_i) = s + (i-1)$ and
- $s \in \{(rn+1)(r+1), (rn+1)(r+1)+n, \ldots, (rn+1)(r+1)+rn\}$.

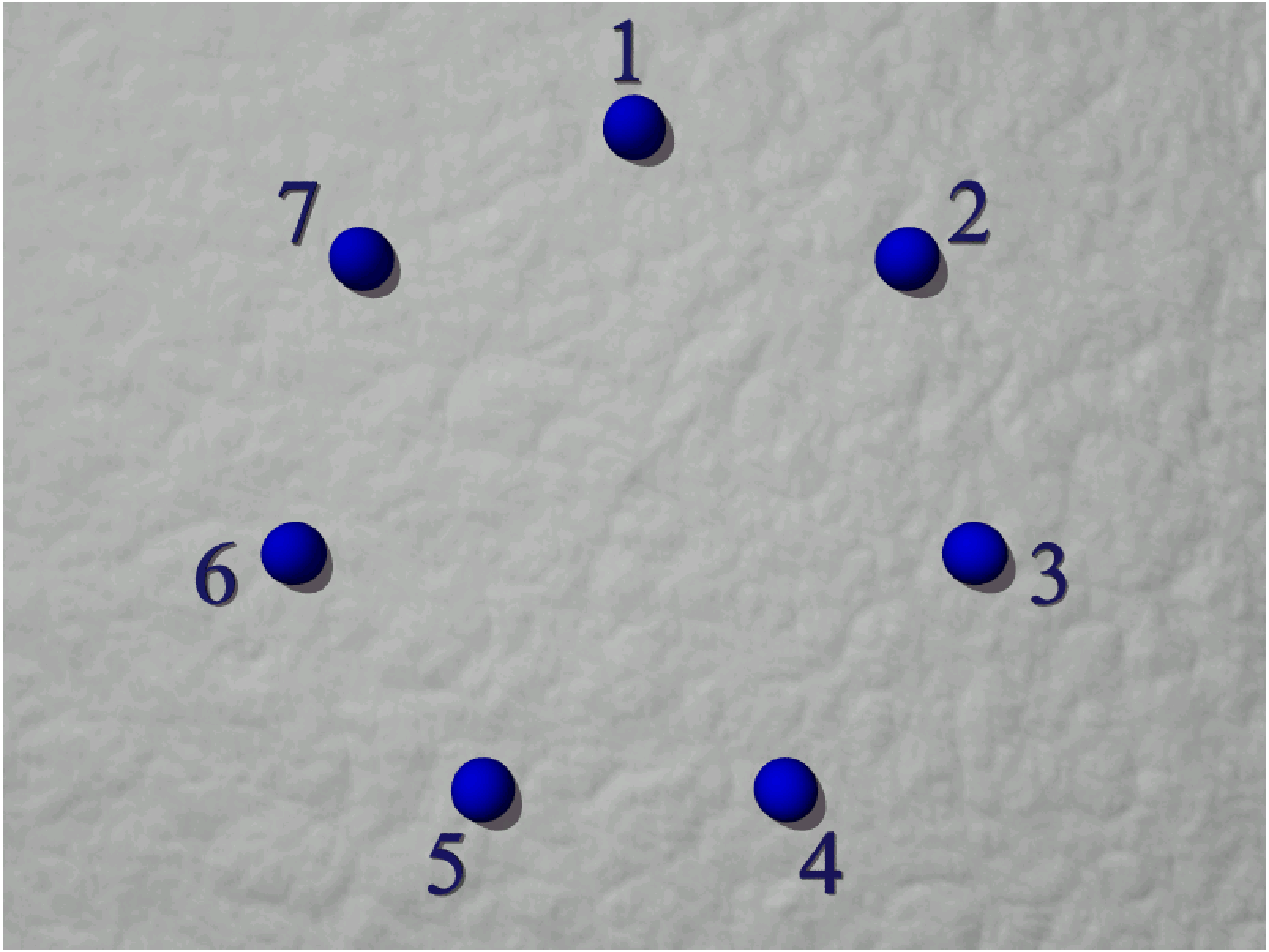
We can pick the weights of vertices arbitrarily.

Example



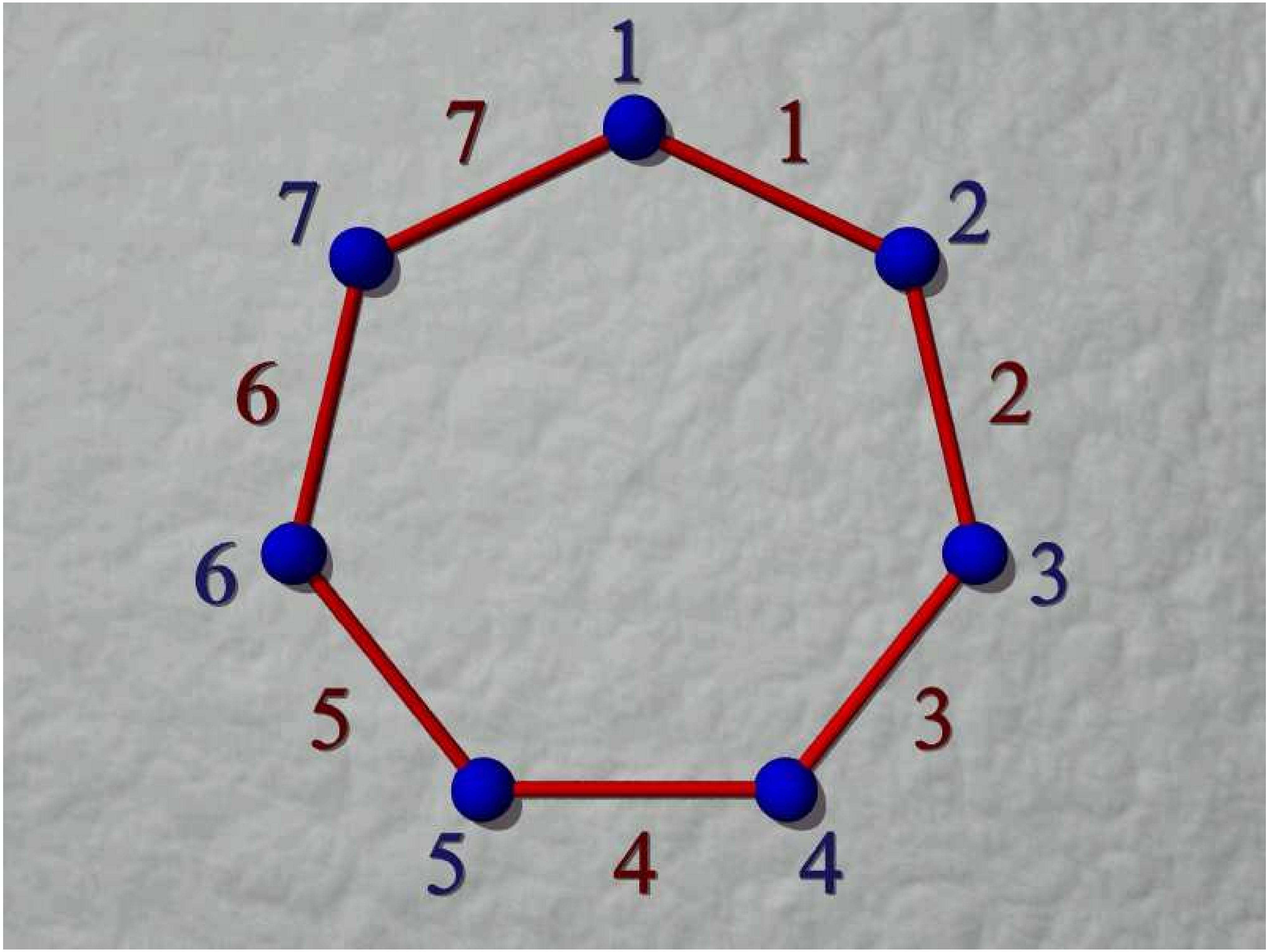
4-regular graph on 7 vertices

Example



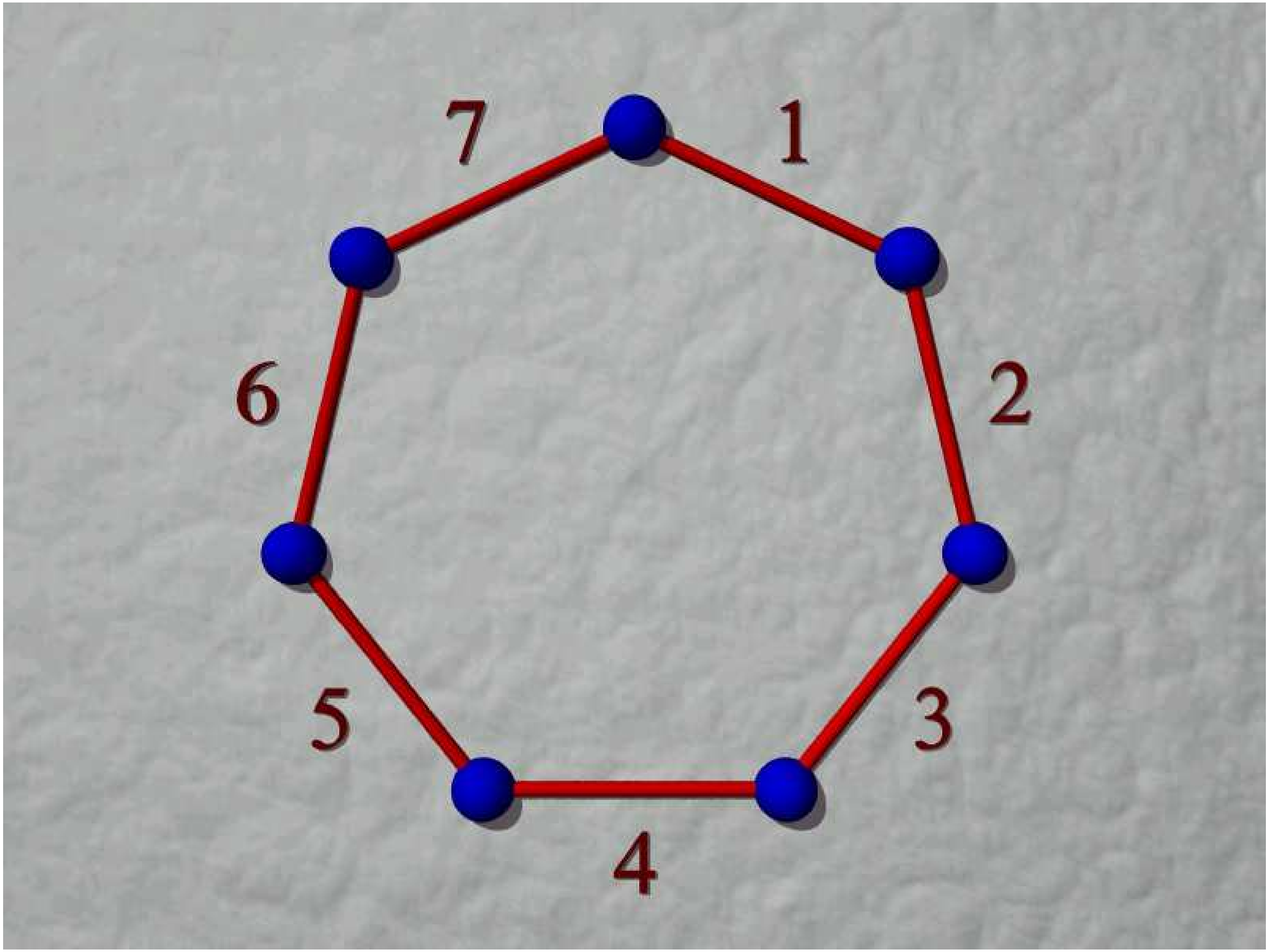
VAMT labeling of $7K_1$

Example (continued) (4/8)



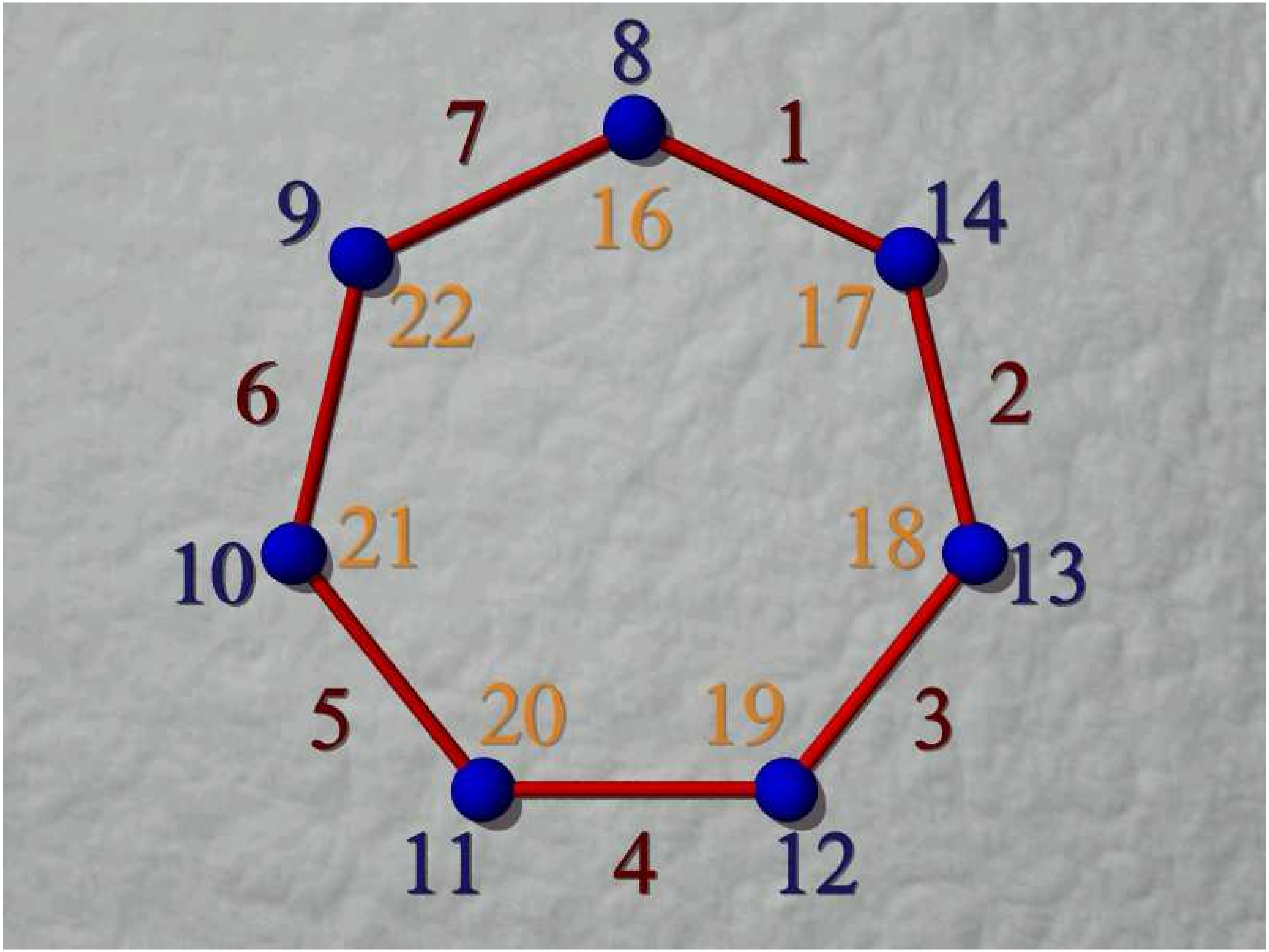
The vertex labels move to the "outgoing" edges

Example (continued) (5/8)



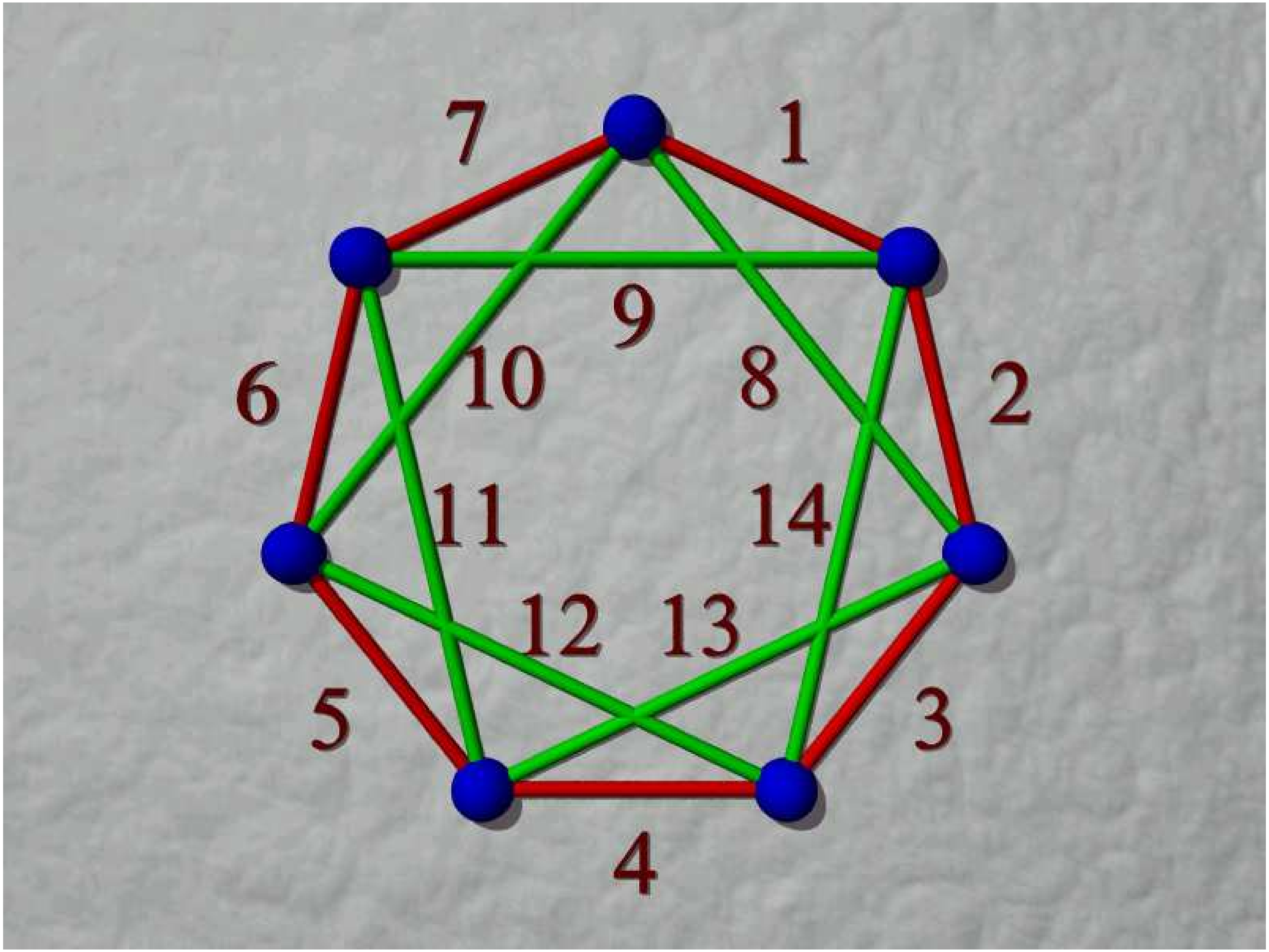
The vertex labels REALLY move to the new edges

Example (continued) (6/8)



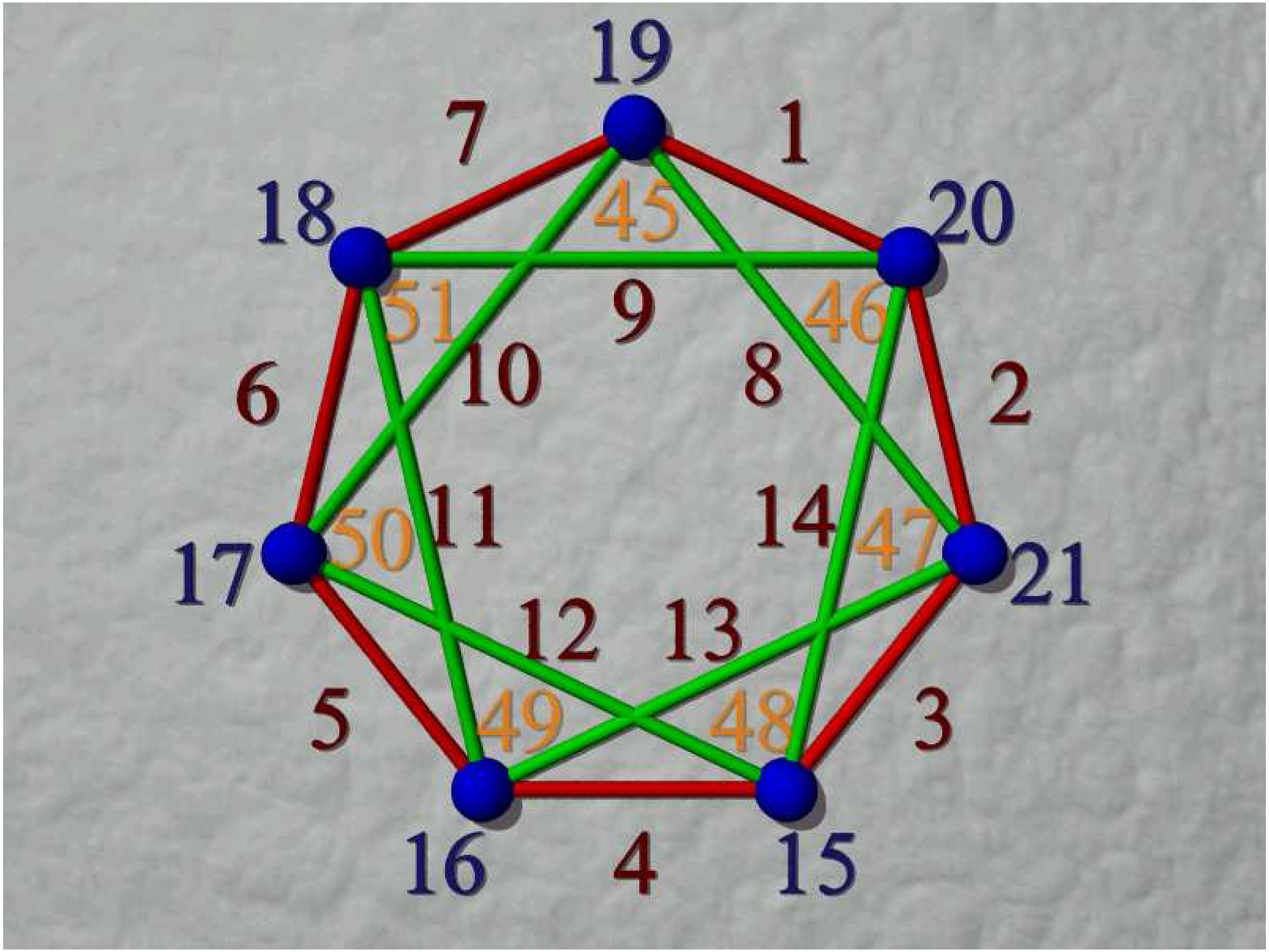
New vertex labels according the incoming edges

Example (continued) (7/8)



The vertex labels move to the "outgoing" edges

Example (continued) (8/8)



New vertex labels according the incoming edges

Results on VMT labelings

- Theorem
- Example
- Further recent results
- VMT from VAMT

Theorem

Let G be a $2r$-regular graph. Let G have a 2-regular factor with a VMT labeling such that vertex labels are consecutive integers. Then G is VMT.

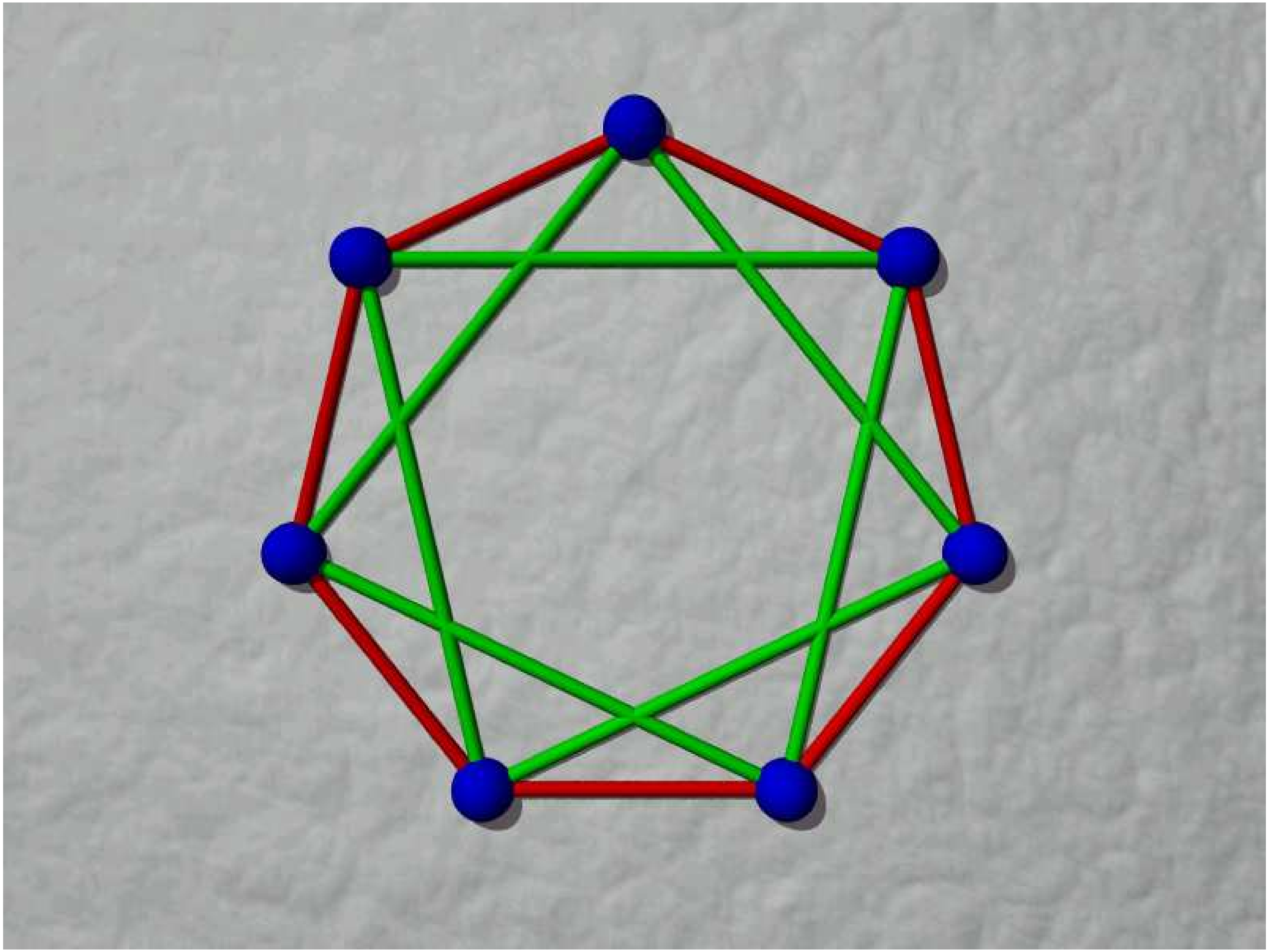
The proof is constructive.
Idea based on Kotzig arrays and Petersen Theorem.

Petersen Theorem

Let G be a $2r$-regular graph.
Then there exists a 2-factor in G.

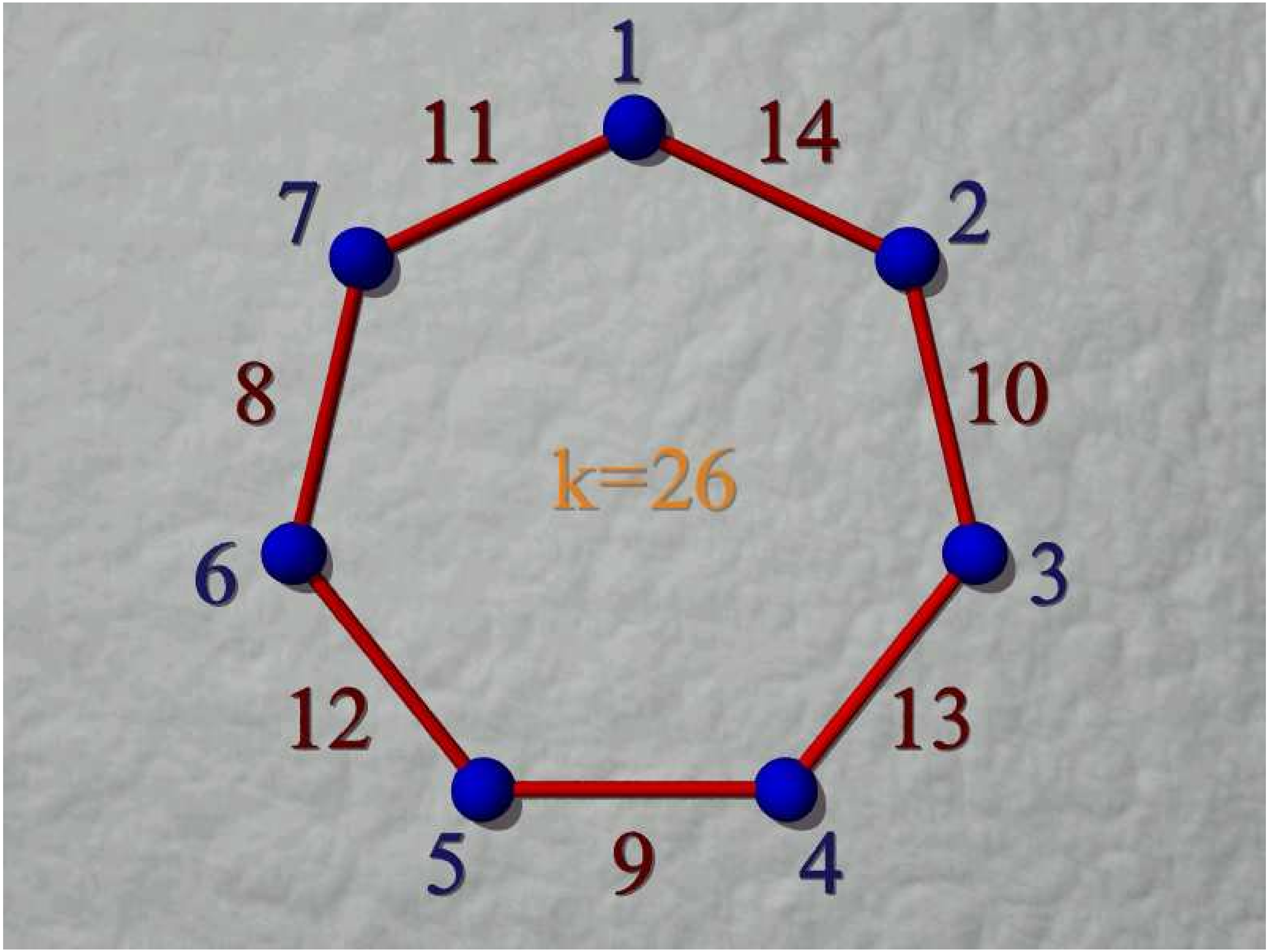
There can be no VMT on 2n vertices with vertices labeled by consecutive integers :o(

Example (2/8)

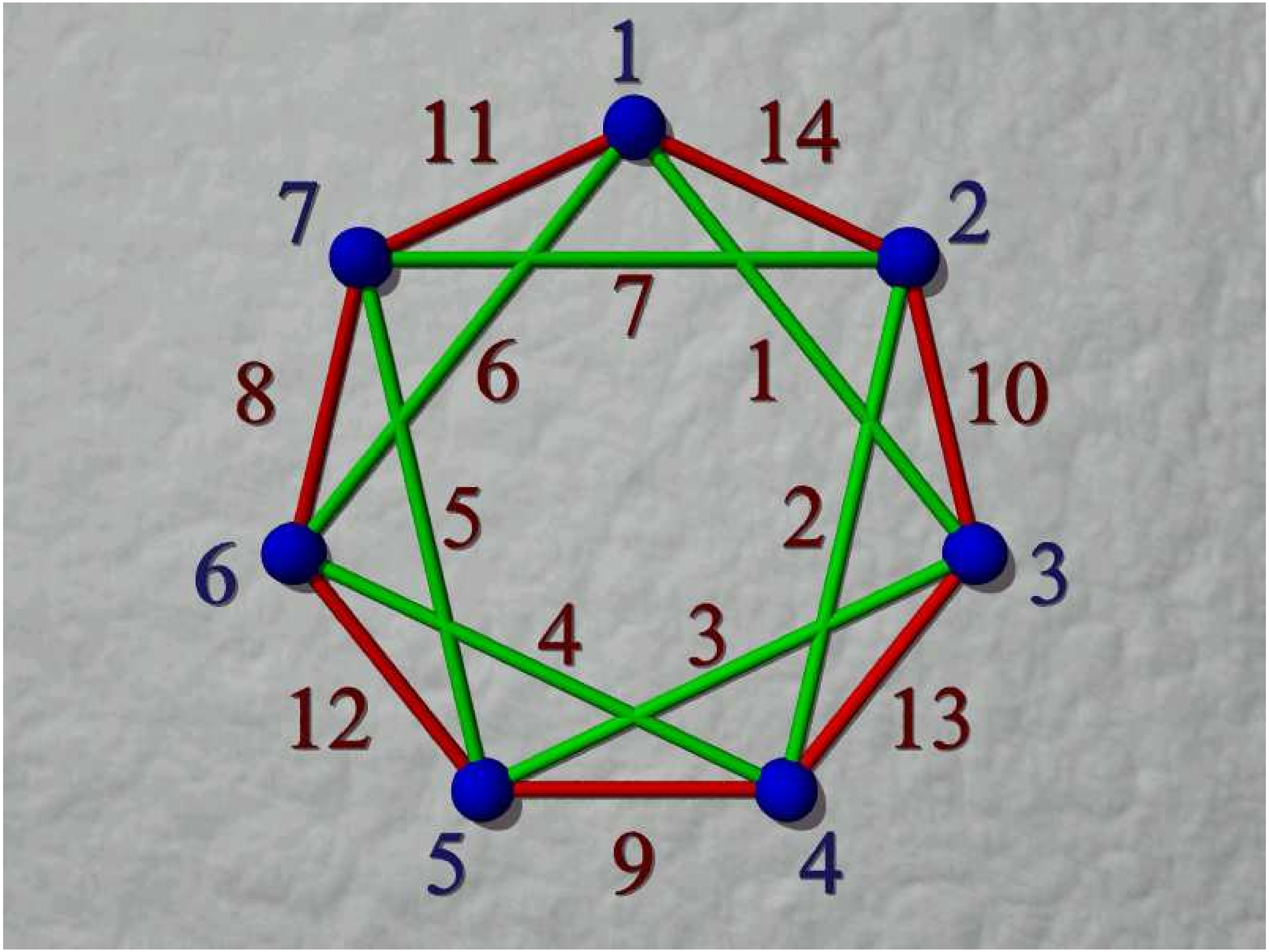


4-regular graph on 7 vertices

Example

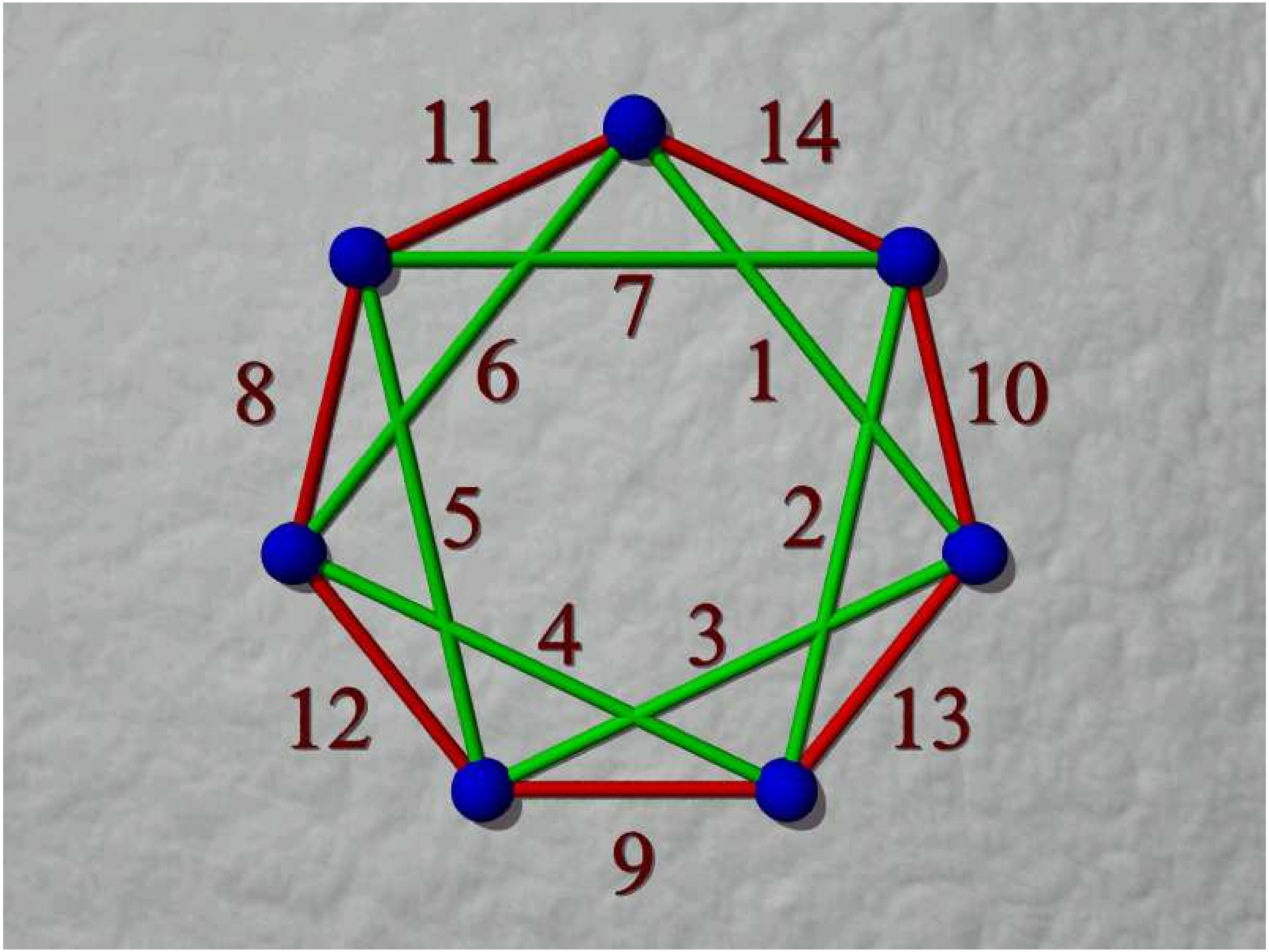


VMT labeling of C_7



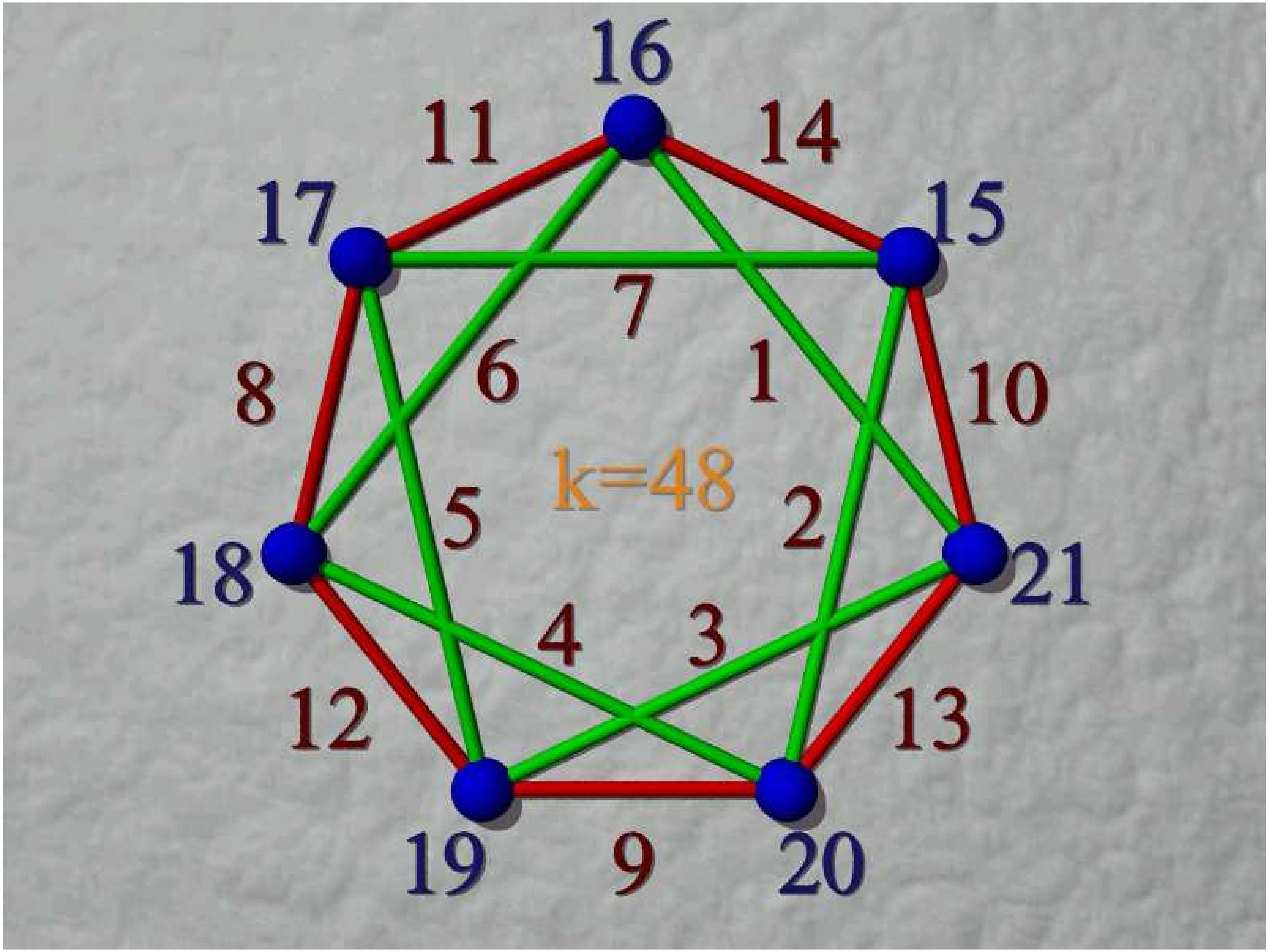
The vertex labels move to the "outgoing" edges

Example (continued) (5/8)



The vertex labels REALLY move to the new edges

Example (continued) (6/8)



New vertex labels according the incoming edges

Corollary

Let G be a $2r$-regular graph, $|V(G)|$ is odd, which has a Hamiltonian cycle. Then G is VMT.

Theorem

Let G be a $(2r+s)$-regular graph. Let there be an s-regular factor of G which has a VMT labeling such that vertex labels are consecutive integers. Then G is VMT.

Open problems

- Find a VMT labeling of any 2-regular graph.
- Find a VMT labeling of C_{2m} x C_{2n}.
- Find (s,d)-VAMT labelings of cyles for d=3,4,5
- Find other methods besides decompositions and Kotzig arrays for finding magic-type labelings of large graphs using magic-type labelings of small graphs?

References

- M. Baca, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae Graph Theory,* **23** (2003), 67-83.
- M. Baca, M. Miller, and Slamin, Every generalized Petersen graph has a vertex-magic total labeling, *Int. J. Comput. Math.,* 11th Australasian Workshop on Combinatorial Algorithms (Hunter Valley, 2000), **79** (2002), 1259-1263.
- J.A. MacDougall, M. Miller, Slamin, and W. D. Wallis, Vertex-Magic total labelings of graphs, *Util. Math.,* **61** (2001), 3-21.
- J. A. MacDougall, M. Miller, Slamin, and W.D. Wallis, Problems in magic total labellings, *Proceedings of AWOCA'99* (1999), 19-25.
- D. Froncek, P. Kovar, and T. Kovarova, Vertex magic total labeling of products of cycles, *submitted.*
- J. A. Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics,* **DS 6** (2003).
- P. Kovar, Vertex magic total labeling of products of regular VMT graphs and regular supermagic graphs, *submitted.*
- P. Kovar, Vertex magic total labeling of products of regular VMT graphs and regular class 1 supermagic graphs, *submitted.*
- P. Kovar, On magic graph labelings, *Master Thesis,* University of Minnesota Duluth, (2004).
- T. Kovarova, Magic labelings of graphs, *Master Thesis,* University of Minnesota Duluth, (2004).
- A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.,* **13** (1970), 451-461.
- Y. Lin and M. Miller, Vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl.,* **33** (2001), 68-76.
- D. McQuillan, Vertex-magic cubic graphs, *JCMCC,* **48** (2004), 103-106.
- J. Sedlacek, On magic graphs, *Math. Slov.,* **26** (1976), 329-335.
- B. M. Stewart, Supermagic complete graphs, *Canadian J. Math.,* **19** (1967), 427-438.
- W. D. Wallis, Magic Graphs, *Birkhauser,* Boston MA, (2001).
- W. D. Wallis, Vertex magic labelings of multiple graphs, *Congressus Numerantium,* **152** (2001), 81-83.

The End

Thank you

for your attention