Matroid intersection and 2-walks in tough graphs

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Outline

We overview the following concepts:

- **toughness** and its relation to hamiltonicity,
- **matroids** and simplicial complexes,
- the **matroid intersection theorem**, and
- results on **independent systems of representatives**.

We then interlink these topics by sketching a proof of a result on 2-walks in tough graphs.

Toughness and hamiltonicity

Toughness

A graph G is t-**tough** ($t > 0$ real) if for all $X \subset V(G)$,

$$G - X \text{ has } \leq |X|/t \text{ components}$$

(whenever it is disconnected)

Toughness $\tau(G)$ of G: maximum t such that G is t-tough (∞ if G is complete)

Observe: t-tough graphs are $2t$-connected

And: Every 1-tough graph has a 1-factor (directly from Tutte's 1-factor theorem)

Theorem 1 (Enomoto et al., 1985) *Every k-tough graph has a k-factor.*

Toughness and hamiltonicity

G hamiltonian $\implies$ G is 1-tough

Conjecture 2 (Chvátal) *There is t_0 such that any t_0-tough graph is hamiltonian.*

Chvátal conjectured $t_0 = 2$ would do, but non-hamiltonian graphs of toughness $\approx 9/4$ were found by Bauer et al. (2000)

There are positive results for chordal graphs:

- 18-tough chordal graphs are hamiltonian (Chen et al. 1998)
- there are $(7/4 - \varepsilon)$-tough chordal non-hamiltonian graphs (Bauer et al. 2000)
- $(1+\varepsilon)$-tough chordal planar graphs are hamiltonian (Böhme, Harant and Tkáč 1999), 1-tough is not enough

k-walks and k-trees

k-walks and k-trees generalize Hamilton cycles and paths, respectively:

k-walk: a closed spanning walk visiting every vertex at most k times

k-tree: a spanning tree of maximum degree at most k

One easily has

$$(k-1)\text{-walk} \implies k\text{-tree} \implies k\text{-walk}$$

Theorem 3 (Win, 1989) *For $k \geq 3$, every $\frac{1}{k-2}$-tough graph has a k-tree.*

Conjecture 4 (Jackson and Wormald) *For $k \geq 2$, every $\frac{1}{k-1}$-tough graph has a k-walk.*

2-walks

For $k = 2$, Jackson and Wormald's conjecture states that every 1-tough graph has a 2-walk.

There are nearly $\frac{17}{24}$-tough graphs with no 2-walk; a finite upper bound is available:

Theorem 5 (Ellingham and Zha) *Every 4-tough graph has a 2-walk.*

Method:

- find a 2-factor F (possible as k-tough graphs have k-factors)
- then show that there is a matching interconnecting all the components of F

F-connecting matchings

G is a graph, F a disconnected spanning subgraph

A matching $T \subset E(G)$ is an **F-connecting matching** if $T \cup F$ is connected

When does an F-connecting matching exist in G?

Idea:

- contract each component of F to obtain G'
- look for a spanning tree T in G'
- but the pairs of edges incident in G are *incompatible* (T must correspond to a matching in G)

So we need a spanning tree in G' with *local constraints*

Colorful spanning trees

This is reminiscent of *colorful spanning trees*:

G is a a graph; edges partitioned into $E_1, \ldots, E_k$ (**colors**)

A spanning tree T of G is **colorful** if it intersects each color in ≤ 1 edge

Theorem 6 *G has a colorful spanning tree if and only if the removal of any t colors leaves a graph with at most $t+1$ components.*

A proof uses the matroid intersection theorem. . .

Part 2

Matroids and complexes

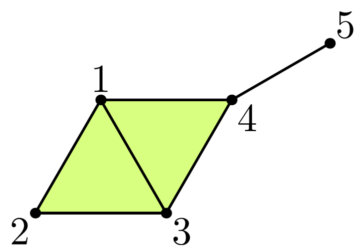
Complexes

(Simplicial) complex K on a set V: a set system on V that is *hereditary*:

$$\text{if} \quad A \subset B \in K, \quad \text{then} \quad A \in K.$$

a complex K has an associated topological space $|K|$ (the **polyhedron** of K)

Example: if K consists of $\{1,2,3\}$, $\{1,3,4\}$, $\{4,5\}$ and all their subsets, then $|K|$ is:



Face of K = set from K

The **induced subcomplex** $K[W]$ on $W \subset V$ consists of all the $A \in K$ with $A \subset W$

Matroids

Matroid: a complex M such that for each X, all the maximal faces of $M[X]$ have the same dimension, the **rank** $r_M(X)$ of X

Independent set of M = face

The maximal faces of M are its **bases**; their dimension is $r(M)$

Common classes of matroids include:

- graphic matroids
- representable matroids
- transversal matroids

Graphic matroids

$G = (V, E)$ is a graph

The **cycle matroid** of G: matroid $M(G)$ on E, the independent sets are edge sets of forests

Example: $G = K_3$ with edges a, b, c: the bases are $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$

If G is connected, then the bases of $M(G)$ are exactly spanning trees

Transversal matroids

$S = (A_1, \ldots, A_k)$ is a set system on V

Transversal (system of distinct representatives) of S: a set $\{a_1, \ldots, a_k\}$ with $a_i \in A_i$ (and the a_i's distinct)

Partial transversal of S = transversal of some $\{A_i : i \in I\}$ where $I \subset \{1, \ldots, k\}$

All partial transversals of S form a matroid on V — a **transversal matroid**

Matroid intersection

The Matroid intersection theorem

The Matroid intersection theorem is this celebrated result of Edmonds:

Theorem 7 *Let M, N be matroids on the same ground set E. If, for every $X \subseteq E$,*

$$r_M(X) + r_N(E - X) \geq k,$$

then M and N have a common k-element independent set.

Application: König's theorem

G is a graph

the **matching number** $\nu(G)$ = size of a largest matching

the **vertex cover number** $\tau(G)$ = minimum number of vertices needed to *cover* all edges

Observe: $\nu(G) \leq \tau(G)$, inequality strict e.g. for K_3

Theorem 8 (König) *If G is bipartite, then $\nu(G) = \tau(G)$.*

Application: Colorful spanning trees

G is a a graph; edges partitioned into colors $E_1, \ldots, E_k$

A spanning tree T of G is **colorful** if it intersects each color in ≤ 1 edge

Theorem 9 *G has a colorful spanning tree if and only if the removal of any t colors leaves a graph with at most $t + 1$ components.*

Other applications

Theorem 10 (Rado) *Let M be a matroid on E. A set system $\{A_i : i \in I\}$ on E has a transversal that is independent in M if and only if*

$$r_M(\bigcup_{i \in X} A_i) \geq |X|$$

for all $X \subset E$.

Observe: Hall's theorem is a corollary

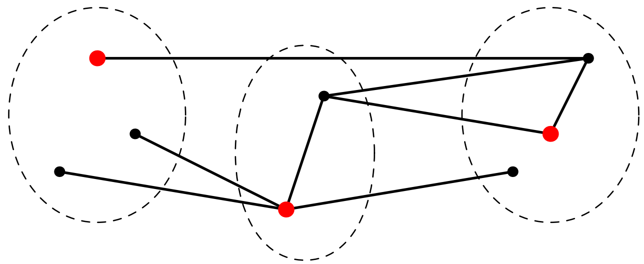
Another corollary is a theorem of Ford and Fulkerson saying when two set systems on the same ground set have a common transversal

Independent systems of representatives

Independent systems of representatives

Throughout: G is a graph, $(P_1, \ldots, P_m)$ a partition of $V(G)$

Independent system of representatives (ISR): an independent set $S \subset V(G)$ with $|S \cap P_i| = 1$ for all i



When does G have an ISR?

Haxell's theorem

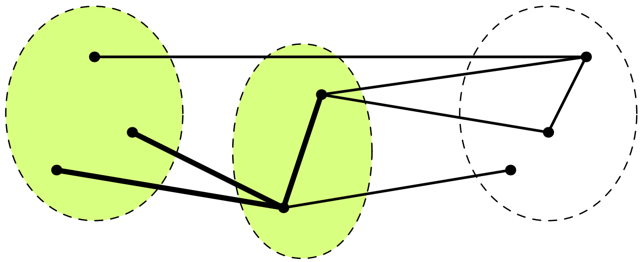
$D \subset V(G)$ is **totally dominating** if every vertex $v \in V(G)$ has a neighbor in D ($v \in D$ is not enough)

total domination number $\tilde{\gamma}(G)$: the size of a smallest totally dominating set

Theorem 11 (Haxell, 1995) *If, for each $I \subseteq \{1, \ldots, m\}$,*

$$\tilde{\gamma}\Big(G\Big[\bigcup_{i \in I} P_i\Big]\Big) \geq 2|I| - 1,$$

then an ISR exists.

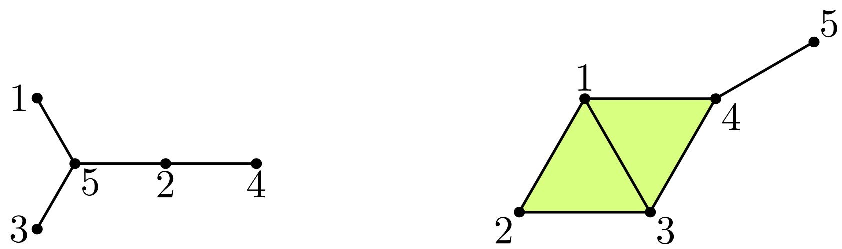


Complexes of graphs

$G = (V, E)$ is a graph

Independence complex $I(G)$ of G: complex on V, faces = independent sets of vertices

Example: A graph and its independence complex (clearly not a matroid!):



Matching complex of G: on E, faces = matchings in G

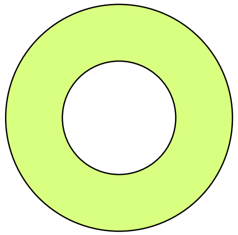
Observe: The matching complex is just $I(L(G))$

Connectivity

A topological space X is **n-connected** if every continuous map from the k-sphere S^k to X can be extended to a map from the $(k+1)$-ball B^{k+1} to X, for all $k \leq n$.

Connectivity $\kappa(X)$: maximum n such that X is n-connected

This space has connectivity 0:



Informally: n-connectedness means 'no holes up to dimension n'

Extensions of Haxell's theorem

(1) Colorful faces

independent transversal $=$ *transversal that is a face of* $I(G)$

Meshulam: Replace $I(G)$ by an arbitrary complex on V

When does a set system have a transversal that is a face of a complex K*?*

(2) Faces independent in a matroid

transversal $=$ *basis of the transversal matroid* T defined by the colors P_i

Aharoni and Berger: Replace T by an arbitrary matroid

When does a matroid have a basis that is a face of a complex K*?*

Colorful faces

Given a graph G, any partition $(P_1, \ldots, P_m)$ of $V(G)$ induces a coloring of the points of $I(G)$; an ISR is then just a **colorful face**

General problem Find a sufficient condition for the existence of a colorful face in a given complex with colored points.

Theorem 12 (Meshulam, 2001) *A complex K with points partitioned into color sets $P_1, \ldots, P_m$ contains a colorful face whenever for each $I \subseteq \{1, \ldots, m\}$, the complex*

$$K\Big[\bigcup_{i \in I} P_i\Big]$$

is $(|I| - 2)$-connected.

Colorful faces and Haxell's theorem

The connectivity of $I(G)$ is related to domination:

$$\kappa(|I(G)|) \geq \frac{\tilde{\gamma}(G)}{2} - 2$$

($+$ other lower bounds of this sort)

Using the above, Meshulam's result specializes to Haxell's theorem.

Aharoni and Berger's theorem

This recent result of Aharoni and Berger generalizes

- the theorems on ISRs and colorful simplices,
- one direction of the Matroid intersection theorem:

Theorem 13 (Aharoni and Berger, 2004) *Let M be a matroid and K a complex on the same ground set E. If, for every $X \subseteq E$,*

$$r_M(X) + \kappa(|K[E - X]|) \geq r(M) - 2,$$

then M has a basis belonging to K.

The connection relies on the following fact: If M is a matroid, then the connectivity of $|M|$ equals $r(M) - 2$.

Putting it together

Back to F-connecting matchings

F is a spanning disconnected subgraph of G, we are looking for an F-connecting matching

Let G' be obtained from G by contracting each component of F

We apply the theorem as follows:

- M is the cycle matroid of G'
- K is the matching complex of $G - E(F)$

The connectivity of K (and subcomplexes) is estimated using a variant of the domination number: $\gamma^{\vee}(H)$ is the minimum number of paths of length 2 dominating all edges of H

Notation:

$\omega(H)$ = the number of components of H

H_X = the spanning subgraph of H with edge set X

Theorem 14 *If for each* $X \subseteq E(G)$*,*

$$\gamma^{\vee}(G_X) \geq \omega(G - X) - 1,$$

then G *contains an* F*-connecting matching.*

For the 2-walk problem, this yields:

Corollary 15 *Every* $(3 + \frac{9}{k-3})$*-tough graph of girth* $\geq k$ *has a* 2*-walk.*