DECOMPOSITIONS OF GRAPHS

Tomáš Vetrík

Faculty of Civil Engineering

Slovak University of Technology

Bratislava, Slovakia

G – graph

$V(G)$ – its vertex set

$E(G)$ – its edge set

$F_1, F_2, ..., F_m$ – a set of factors of G

d(G) – diameter of G

The study of decompositions of complete graphs into factors with given diameters was initiated by Bosák, Rosa and Znám.

Theorem 1 *If complete graph K_n is decomposable into m factors $F_1, F_2, ..., F_m$ with prescribed diameters $d_1, d_2, ..., d_m$, then for every $n' > n$ the complete graph $K_{n'}$ is also decomposable into such factors.*

$F(d_1, d_2, ..., d_m)$ – the smallest positive integer n such that the graph K_n can be decomposed into m factors with prescribed diameters $d_1, d_2, ..., d_m$.

If all the factors have the same diameter d, we write $F(d_1, d_2, ..., d_m) = F_m(d)$.

n is admissible with respect to m
if m divides $n(n-1)/2$

$G_m(d)$ – the smallest positive integer n such that the graph K_n can be decomposed into m isomorphic factors of diameter d.

$H_m(d)$ – the smallest admissible integer n such that for all admissible integers $n' \geq n$, the complete graph $K_{n'}$ is also decomposable into such isomorphic factors of diameter d.

It is obvious that

$$F_m(d) \leq G_m(d) \leq H_m(d).$$

Conjecture (Kotzig and Rosa)
For every $m, d \geq 2$ it holds that

$$G_m(d) = H_m(d).$$

The truth of the conjecture has been verified For $m = 2$ and for $m = 3$ if $d = 3, 4, 5, 6$.

DECOMPOSITIONS OF COMPLETE GRAPHS INTO 3 FACTORS

Let us assume that the diameters d_1, d_2, d_3 satisfy

(1) $$d_1 \leq d_2 \leq d_3 < \infty.$$

Theorem 2 (Bosák, Rosa and Znám)
Let $d_1 \geq 5$*. Then* $F(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8$*.*

Theorem 3 (Hrnčiar) *Let* $d_1 > 65$*.*
Then $F(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8$*.*

$$d_1 = 2$$

Theorem 4 (Bosák, Rosa and Znám)
Let $d_2 \geq 2$*. If* $d_3 \geq 7$*, then* $F(2, d_2, d_3) = d_3 + 1$
except $F(2, 2, 7)$*.*
$F(2,6,6) = F(2,5,6) = F(2,5,5) = F(2,4,6) = F(2,4,5) = F(2,4,4) = 7$*;*
$F(2,3,6) = F(2,3,5) = F(2,3,4) = F(2,3,3) = 8$*;*
$F(2,2,7) = F(2,2,6) = F(2,2,5) = 9$*;*
$F(2,2,4) = F(2,2,3) = 10$*;* $12 \leq F(2,2,2) \leq 13$*.*

Theorem 5 (Stacho, Urland) $F(2,2,2) = 13$*.*

$$d_1 = 3$$

Theorem 6 (Palumbíny) *Let* $d_2 \geq 7$*.*
Then $F(3, d_2, d_3) \leq d_2 + d_3 - 6$*.*

Theorem 7 (Palumbíny)
$F(3,3,3) = F(3,4,4) = F(3,5,5) = 6$*;*
$F(3,3,4) = F(3,3,5) = F(3,3,6) = F(3,4,5) =$
$F(3,4,6) = F(3,5,6) = F(3,6,6) = 7$*.*
If $3 \leq d_2 \leq 7 \leq d_3$*, then* $F(3, d_2, d_3) = d_3 + 1$*.*
If $d_3 \geq 8$*, then* $F(3, 8, d_3) = d_3 + 2$*.*

Results

The auxiliary result needed in the proof of Theorem 8.

Lemma 1 *Let u, v be two distinct vertices of a graph G with n vertices ($n \geq 5$) and finite diameter d ($d \geq 2$). Let $deg(u) = a$, $deg(v) = b$ and $a + b > n - d + 3$.*

I. If the edge $uv \in G$, then there exists at least $(a + b) - (n - d + 3)$ vertices adjacent to both vertices u and v in graph G.

II. If the edge $uv \notin G$, then there exists at least $(a + b) - (n - d + 2)$ vertices adjacent to both vertices u and v in graph G.

Theorem 8 *Let $d_2 \geq 9$.*
Then $F(3, d_2, d_3) = d_2 + d_3 - 6$.

$$d_1 = 4$$

Theorem 9 (Říhová) *Let* $d_2 \geq 5$, $d_3 \geq 6$. *Then* $F(4, d_2, d_3) \leq d_2 + d_3 - 4$.

Theorem 10 (Říhová)
I. $F(4,4,4) = F(4,5,5) = 6$;
II. Let $d_3 \geq 5$. *Then* $F(4,4,d_3) = d_3 + 1$,
III. Let $d_3 \geq 6$. *Then* $F(4,5,d_3) = d_3 + 1$.

Results

Theorem 11 *Let* $d_2 \geq 6$. *Then* $F(4, d_2, d_3) \leq d_2 + d_3 - 5$.

Corollary 1 *Let* $d_3 \geq 6$. *Then* $F(4, 6, d_3) = d_3 + 1$.

Theorem 12 *Let* $d_2 \geq 7$. *Then* $F(4, d_2, d_3) = d_2 + d_3 - 5$.