SQUARE OF METRICALLY REGULAR GRAPHS

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Let X be a finite set, card $X \geq 2$. For an arbitrary natural number D let $\boldsymbol{R} = \{R_0, R_1, \ldots, R_D\}$ be a system of binary relations on X. A pair $(X,\boldsymbol{R})$ will be called **an association scheme** with D classes if and only if it satisfies the axioms
$A1$ - $A4$:

A1. The system $\boldsymbol{R}$ is a partition of the set X^2 and R_0 is the diagonal relation, i.e. $R_0 = \{(x, x);\ x \in X\}$.

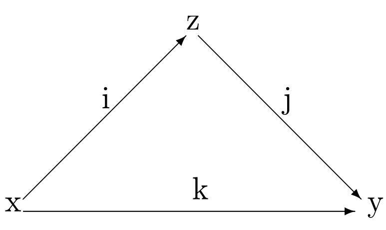
A2. For each $i \in \{0, 1, \ldots, D\}$, it holds $R_i^{-1} \in \boldsymbol{R}$.

A3. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds

$$(x, y) \in R_k \wedge (x_1, y_1) \in R_k \Longrightarrow p_{ij}(x, y) = p_{ij}(x_1, y_1),$$

where

$$p_{ij}(x, y) = |\{z; (x, z) \in R_i \wedge (z, y) \in R_j\}|.$$



Then define

$$p_{ij}^k := p_{ij}(x, y),\ \text{where } (x, y) \in R_k.$$

A4. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds $p_{ij}^k = p_{ji}^k$.

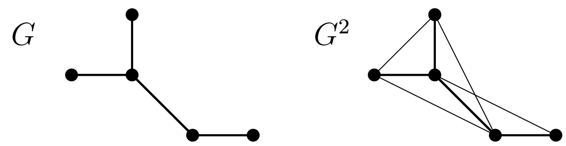
The set X will be called *the carrier* of the association scheme $(X, \boldsymbol{R})$. In particular, $p_{i0}^k = \delta_{ik},\ p_{ij}^0 = v_i\delta_{ij}$, where δ_{ij} is the Kronecker-symbol and $v_i := p_{ii}^0$ and define

$$P_j := (p_{ij}^k),\ 0 \leq i, j, k \leq D.$$

(See E.Bannai, T.Ito. *Algebraic Combinatorics I.* The Bejamin/Cummings Publishing Company, California, 1984 .)

Given an undirected graph $G = (X, E)$ of diameter D we may now define $R_k = \{(x, y); d(x, y) = k\}$, where $d(x, y)$ is the distance from the vertex x to the vertex y in the standard graph metric. If $(X, \boldsymbol{R})$ gives rise to an association scheme, the graph G is called ***metrically regular*** (sometimes also called ***distance regular***) and p^k_{ij} are said to be its ***parameters***. In particular, a metrically regular graph with diameter $D = 2$ is called ***strongly regular***.

Let $G = (X, Y)$ be an undirected graph without loops and multiple edges. ***The second power*** (or ***square*** of G) is the graph $G^2 = (X, E')$ with the same vertex set X and in which mutually different vertices are adjacent if and only if there is at least one path of length 1 or 2 in G between them.



The characteristic polynomial of the adjacency matrix A of a graph G is called the ***characteristic polynomial*** of G and the eigenvalues and the spectrum of A are called the ***eigenvalues*** and the ***spectrum of G***. The greatest eigenvalue of G is called the ***index*** of G.

1 Conditions for metrically regular graphs of diameter D to have their square metrically regular

(The same problem we can find in Brouwer A.E., Cohen A.M., Neumaier A.: *Distance-regular graphs*, p. 151, Springer-Verlag 1989.)

Because $G = (X, E)$ is metrically regular, the pair $(X, \boldsymbol{R})$ forms an association scheme with parameters p^k_{ij}, where

$$\boldsymbol{R} = \{R_0, R_1, \ldots, R_D\}, R_i = \{(x, y); x, y \in X, d(x, y) = i\}.$$

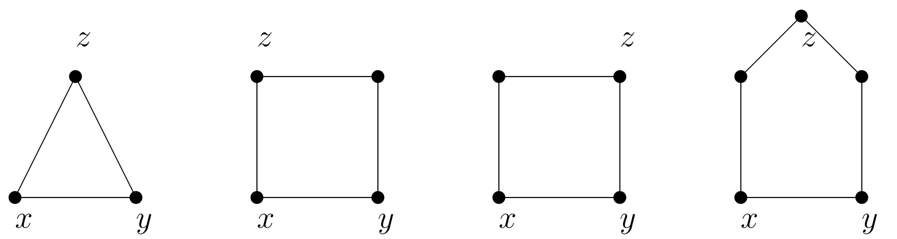
If $G^2 = (X, E')$ is metrically regular then the pair $(X, \boldsymbol{R}')$ forms an association scheme too, where $\boldsymbol{R}' = \{R'_0, R'_1, \ldots, R'_{D'}\}$ and $R'_0 = R_0$, $R'_k = R_{2k-1} \cup R_{2k}$. So, for its parameters ${}^2p^k_{i,j}$ it must hold

$${}^2p^k_{ij} = \sum_{m,n=0}^{1} p^{2k-1}_{2i-m,2j-n} = \sum_{m,n=0}^{1} p^{2k}_{2i-m,2j-n}, \quad 1 \le i, j, k \le D'$$

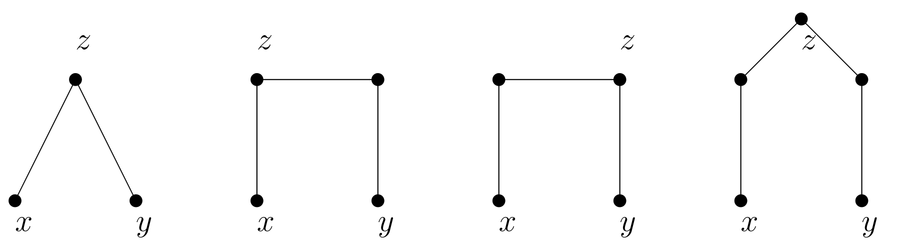
There is the example in the following picture.

$${}^2p^1_{11} =$$

$$= p^1_{11} \qquad + p^1_{12} \qquad + p^1_{21} \qquad + p^1_{22} =$$



$$= p^2_{11} \qquad + p^2_{12} \qquad + p^2_{21} \qquad + p^2_{22}$$



On the other hand, if A denotes the adjacency matrix of the metrically regular graph G and A_2 the adjacency matrix of G^2 it holds

$$A_2 = \frac{1}{p^2_{11}} A^2 + \frac{p^2_{11} - p^1_{11}}{p^2_{11}} A \; - \frac{\lambda_1}{p^2_{11}} I.$$

So, if the eigenvalues of G are $\lambda_1 > \cdots > \lambda_k$ with respective multiplicities $m_1 = 1, m_2, \ldots, m_k$, the eigenvalues of G^2 are in the form

$$\mu_i = \frac{\lambda_i^2 + (p^2_{11} - p^1_{11})\lambda_i - \lambda_1}{p^2_{11}} \tag{1}$$

with multiplicities

$$m'_i = \sum_{j \in M_i} m_j, \;\; M_i = \{j; \mu_j = \mu_i\}.$$

It is proved for metrically regular bipartite graphs of diameter $D \leq 7$ that the spectrum of the metrically regular graph G^2 is in the form

$$S_p(G^2) = \left\{ \begin{array}{lllll} \mu_1, & \mu_2 = \mu_{D+1}, & \mu_3 = \mu_D, & \mu_4 = \mu_{D-1}, & \ldots \\ 1, & m_2 + m_{D+1}, & m_3 + m_D, & m_4 + m_{D-1}, & \ldots \end{array} \right\}.$$

(It would be reasonable to conjecture that this holds for general D). Thus, from (1) we obtain for bipartite graphs:

$$\left(p_{ij}^{k} = 0,\ i + j + k \equiv 1 \pmod 2\right)$$

$$-p_{11}^{2} = \lambda_2 + \lambda_{D+1} = \lambda_3 + \lambda_D = \lambda_4 + \lambda_{D-1} = \cdots .$$

2 Bipartite graphs

D = 3, 4

V.Vetchý. *Metrically regular square of metrically regular bigraphs I*, Arch. Math. Brno, Tomus 27b (1991), 183 - 197.

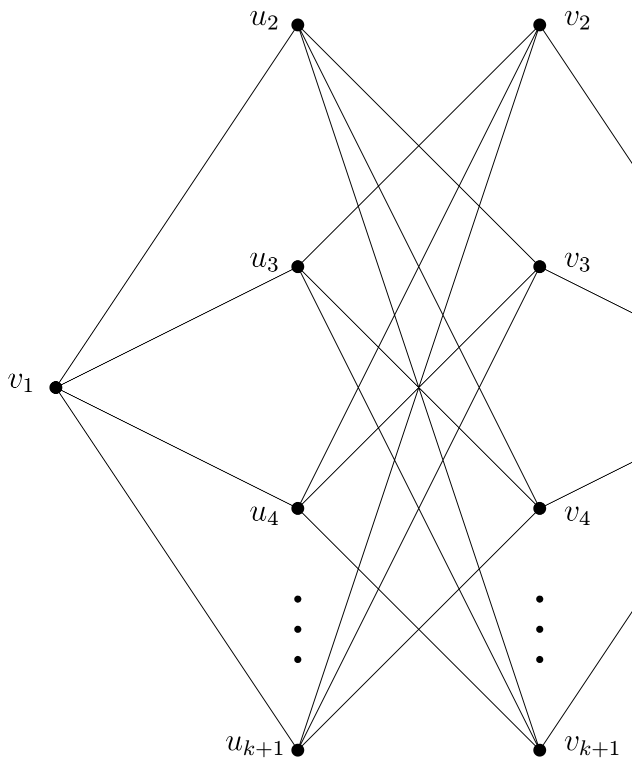
Theorem 1. *Let G be a metrically regular graph with diameter $D = 3$ (4 distinct eigenvalues) and G^2 be strongly regular. Then it holds $\lambda_3 = -1, \lambda_2 > 0$.*

Theorem 2. *For every $k \in N,\ k \geq 2$ there is only one metrically regular bipartite graph $G = (X, E)$ with diameter $D = 3,\ |X| = 2k+2$ so that G^2 is a strongly regular graph. The nonzero parameters of G are the following:*
$p_{10}^{1} = p_{23}^{1} = p_{20}^{2} = p_{13}^{2} = p_{30}^{3} = 1$ $\quad v_0 = v_3 = 1$
$p_{12}^{1} = p_{11}^{2} = p_{22}^{2} = k - 1$ $\quad v_1 = v_2 = k$
$p_{12}^{3} = k$

$$S_p(G) = \left\{ \begin{array}{cccc} k, & 1, & -1, & -k \\ 1, & k, & k, & 1 \end{array} \right\}$$

The construction of $G = \{(X = X_1 \cup X_2, E)\}$:
$X_1 = \{v_1, \ldots, v_{k+1}\}$,
$X_2 = \{u_1, \ldots, u_{k+1}\}$,
$E = \{(v_i, u_j) \mid i, j = 1, 2, \ldots, k+1; i \neq j\}$

The realization is the following graph:



Theorem 3. *There is only one table of parameters of an association scheme so that the corresponding metrically regular bipartite graph of diameter $D = 4$ (5 distinct eigenvalues) has a strongly regular square. The nonzero parameters of G are the following:*

$p^1_{01} = p^2_{20} = p^3_{30} = p^4_{40} = p^3_{14} = p^2_{24} = p^1_{34} = 1$ $\qquad v_0 = v_4 = 1$
$p^2_{11} = p^2_{13} = p^2_{33} = 2$ $\qquad v_1 = v_3 = 4$
$p^1_{12} = p^1_{23} = p^3_{12} = p^3_{23} = 3$ $\qquad v_2 = 6$
$p^2_{22} = p^4_{13} = 4$
$p^4_{22} = 6$ $\qquad S_p(G) = \left\{ \begin{array}{rrrrr} 4, & 2, & 0, & -2, & -4 \\ 1, & 4, & 6, & 4, & 1 \end{array} \right\}$

The realization of this table is the 4-dimensional unit cube.

D = 5

V.Vetchý. *Metrically regular square of metrically regular bigraphs II*, Arch. Math. Brno, Tomus 28 (1992), 17 - 24 .

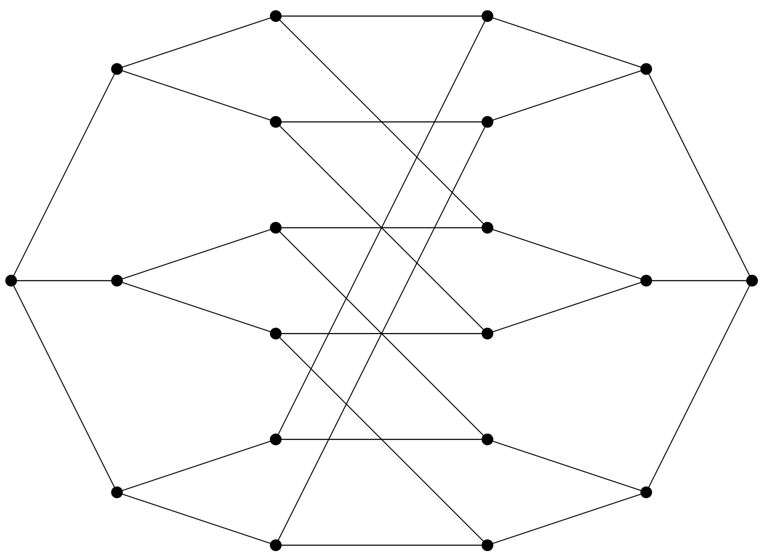
Theorem 4. *There are only four tables of parameters of association schemes so that the corresponding metrically regular bipartite graphs of diameter $D = 5$ (6 distinct eigenvalues) have a metrically regular square. The nonzero parameters of G are the following:*

$p^i_{i0} = p^1_{45} = p^2_{35} = p^3_{25} = p^4_{15} = 1$ $\qquad k = p^2_{11} = p^2_{44} = p^3_{14}$
$p^2_{13} = p^2_{24} = p^3_{12} = p^3_{34} = k+1$ $\qquad 2k = p^1_{12} = p^1_{34} = p^4_{13} = p^4_{24}$
$p^5_{14} = 2k+1,$ $\qquad p^1_{23} = p^4_{22} = p^4_{33} = 2k+2,$ $\qquad v_0 = v_5 = 1$
$p^2_{22} = p^2_{33} = p^3_{23} = 3k$ $\qquad 2k+1 = v_1 = v_4$
$p^5_{23} = 2(2k+1)$ $\qquad 2(2k+1) = v_2 = v_3$

$$S_p(G) = \left\{ \begin{matrix} \pm(2k+1), & \pm(k+1), & \pm 1 \\ 1, & 8 - \frac{12}{k+2}, & 6k - 5 + \frac{12}{k+2} \end{matrix} \right\},$$

$k \in \{1, 2, 4, 10\}$.

In the case $k = 2$ the realization of this table is the 5-dimensional unit cube, for $k = 1$ the realization is the following graph:



D = 6

V.Vetchý. *Metrically regular square of metrically regular bipartite graphs of diameter D=6*, Arch. Math. Brno Tomus 29 (1993), 29 - 38.

Theorem 5. *There is only one table of parameters of an association scheme with 6 classes so that the corresponding metrically regular bipartite graph of diameter $D = 6$ (7 distinct eigenvalues) has a metrically regular square.*

$p^1_{10} = 1$	$p^2_{20} = 1$	$p^3_{30} = 1$	$p^4_{40} = 1$	$p^5_{50} = 1$	$p^6_{60} = 1$
$p^1_{12} = 5$	$p^2_{11} = 2$	$p^3_{12} = 3$	$p^4_{13} = 4$	$p^5_{14} = 5$	$p^6_{15} = 6$
$p^1_{23} = 10$	$p^2_{13} = 4$	$p^3_{14} = 3$	$p^4_{15} = 2$	$p^5_{16} = 1$	$p^6_{24} = 15$
$p^1_{34} = 10$	$p^2_{22} = 8$	$p^3_{23} = 9$	$p^4_{22} = 6$	$p^5_{23} = 10$	$p^6_{33} = 20 = v_3$
$p^1_{45} = 5$	$p^2_{24} = 6$	$p^3_{25} = 3$	$p^4_{24} = 8$	$p^5_{25} = 5$	$\lambda_6 = -4$
$p^1_{56} = 1$	$p^2_{33} = 12$	$p^3_{34} = 9$	$p^4_{26} = 1$	$p^5_{34} = 10$	$m_6 = 6$
$\lambda_1 = 6$	$p^2_{35} = 4$	$p^3_{36} = 1$	$p^4_{33} = 12$	$\lambda_4 = 0$	$\lambda_7 = -6$
$m_1 = 1$	$p^2_{44} = 8$	$p^3_{45} = 3$	$p^4_{35} = 4$	$m_4 = 20$	$m_7 = 1 = v_6$
$\lambda_2 = 4$	$p^2_{46} = 1$	$\lambda_3 = 2$	$p^4_{44} = 6$	$\lambda_5 = -2$	$v_1 = 6 = v_5$
$m_2 = 6$	$p^2_{55} = 2$	$m_3 = 15$	$v_0 = 1$	$m_5 = 15$	$v_2 = 15 = v_4$

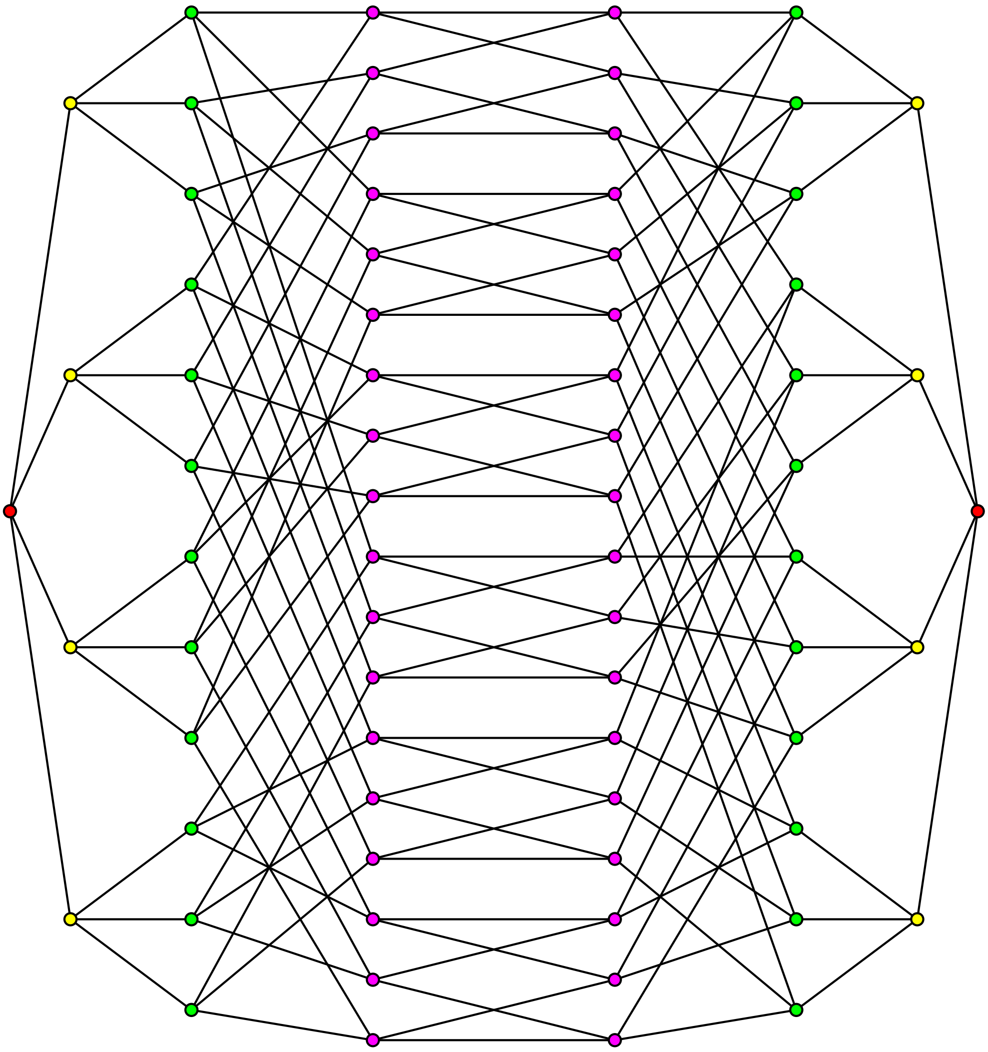
The realization of this table is the 6*-dimensional unit cube.*

D = 7

V.Vetchý. *Metrically regular square of metrically regular bipartite graphs of diameter D=7*, to appear.

Theorem 6. *There are only two tables of the parameters of association schemes so that the corresponding metrically regular bipartite graphs with 8 distinct eigenvalues (diameter $D = 7$) have the metrically regular square.*

The realization of the case $p^2_{11} = 2$ is the 7-dimensional unit cube. The realization of the case $p^2_{11} = 1$ is the following graph:



3 Conjecture

With respect to Theorems 1.– 6. it would be reasonable to conjecture:

There is only one table of parameters of an association scheme with $2k$ *classes* ($k \geq 2$) *so that the corresponding metrically regular bipartite graph of diameter* $D = 2k$ *has a metrically regular square. The realization of this table is the* $2k$*-dimensional unit cube.*

4 General case for D=3

In this case follows from the conditions to be the square G^2 strongly regular for the spectrum of G

$$S_p(G) = \left\{ \begin{array}{cccc} \lambda_1 = r, & \lambda_2, & -1, & \lambda_4 \\ 1, & m_2, & m_3, & m_4 \end{array} \right\}$$

$$\lambda_{2,4} = \frac{p^1_{11} - p^2_{11} \pm \sqrt{(p^1_{11} - p^2_{11})^2 + 4(\lambda_1 + p^2_{11}p^3_{13})}}{2}$$

A. $\sqrt{(p^1_{11} - p^2_{11})^2 + 4(\lambda_1 + p^2_{11}p^3_{13})} = t \in \mathbf{N}$

This implies

$$\lambda_{2,4} = \frac{a \pm c}{2}, \qquad c \in \mathbb{N}.$$

$$m_2(c+a) = m_4(c-a) + 2t, \qquad t \in \mathbb{N}, t > 1,$$

$$m_3 = \lambda_1 + t, \qquad t \in \mathbb{N}, t > 1.$$

From the condition for the traces of A^2 and A^3 of the adjacency matrix of G we obtain

$$4cp^2_{11}p^3_{13}m_4 = t(c + a \ + 2)(2\lambda_1 - c - a)$$

$$2cm_4p^3_{12}(2\lambda_1 - c - a \ + 2p^2_{11}p^3_{13}) = v_2(\lambda_1 + 1)(c + a)(2\lambda_1 - c - a).$$

In this case we get for

$\lambda_1 = 3$ - only one graph - *bipartite graph on 8 vertices*
$\lambda_1 = 4$ - three graphs - *bipartite graph on 10 vertices*
- *2 graphs on 10 and 35 vertices*
$\lambda_1 = 5$ - two graph - *bipartite graph on 12 vertices*
- *graph on 36 vertices*

B. $\sqrt{(p^1_{11} - p^2_{11})^2 + 4(\lambda_1 + p^2_{11}p^3_{13})} \notin \mathbf{N}$

$$m_2 = m_4$$

For the parameters we obtain the following conditions

$$(p^1_{11} - p^2_{11})(\lambda_1 - 1 - p^1_{11} + p^2_{11}) = 2p^2_{11}p^3_{13}$$

$$m_2(p^1_{11} - p^2_{11} + 2)p^2_{11}(\lambda_1 - p^3_{13}) = \lambda_1(\lambda_1 + 1)p^1_{12}$$

According to these conditions we can consider two cases $p^1_{11} = p^2_{11}$ or $p^1_{11} > p^2_{11}$.

1. $\mathbf{p^1_{11} = p^2_{11}}$.

We obtain the following table of the nonzero parameters of the association scheme such that there can exist metrically regular graphs G realizing this table with spectrum $S_p(G)$:

$$
\begin{array}{lll}
p^1_{10} = 1 & p^2_{20} = 1 & p^3_{30} = 1 \qquad (2)\\
p^1_{11} = p^2_{11} & p^2_{11} = p^2_{11} & p^3_{12} = 1 + p^2_{11}(1+v_3) \qquad (3)\\
p^1_{12} = v_3 p^2_{11} & p^2_{12} = v_3 p^2_{11} & p^3_{22} = [1 + p^2_{11}(1+v_3)](v_3 - 1) \qquad (4)\\
p^1_{22} = v_3^2 p^2_{11} & p^2_{13} = 1 & p^3_{33} = v_3 - 1 \qquad (5)\\
p^1_{23} = v_3 & p^2_{22} = v_3^2 p^2_{11} & v_1 = 1 + p^2_{11}(1+v_3) \qquad (6)\\
v_0 = 1 & p^2_{23} = v_3 - 1 & v_2 = v_3[1 + p^2_{11}(1+v_3)] \qquad (7)
\end{array}
$$

$$
S_p(G) =
$$

$$
= \begin{pmatrix} 1 + p^2_{11}(1+v_3) & \sqrt{1 + p^2_{11}(1+v_3)} & -1 & -\sqrt{1 + p^2_{11}(1+v_3)} \\ 1 & v_3 + p^2_{11}\frac{v_3(v_3+1)}{2} & 1 + p^2_{11}(1+v_3) & v_3 + p^2_{11}\frac{v_3(v_3+1)}{2} \end{pmatrix},
$$

$$
v_3 \in \mathbb{N}
$$

The least example is on 12 vertices.