Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

# Small critical snarks and their generalizations

Master's Thesis

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## ZADANIE ZÁVEREČNEJ PRÁCE

**Meno a priezvisko študenta:** Bc. Jozef Rajník
**Študijný program:** informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
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*Malé kritické snarky a ich zovšeobecnenia*

**Anotácia:** Práca vychádza zo zaujímavých pozorovaní bakalárskej práce J. Rajníka. Existujúce výsledky budú rozšírené o analýzu kritických snarkov, ktoré nie sú bikritické, a ireducibilných snarkov s menšou cyklickou súvislosťou. Objavené konštrukčné princípy využijeme na vytvorenie nových tried kritických a ireducibilných snarkov. Dôkazy ireducibility budú využívať nové metódy, jednak výpočtové (redukcia na problém splniteľnosti), jednak teoretické.

**Cieľ:** 1. Vyvinúť nové metódy (výpočtové aj teoretické) na dokazovanie nezafarbiteľnosti a ireducibility snarkov.
2. Skonštruovať nové triedy ireducibilných snarkov (azda sa podarí vytvoriť aj nejaké cyklicky 6-súvislé).

**Vedúci:** RNDr. Ján Mazák, PhD.
**Katedra:** FMFI.KI - Katedra informatiky
**Vedúci katedry:** prof. RNDr. Martin Škoviera, PhD.

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# Abstract

Snarks are non-trivial bridgeless cubic graphs that are not 3-edge-colourable. This thesis extends the analysis of the structure of small bicritical snarks by the same author to strictly critical snarks. This analysis divides examined snarks into several infinite classes. We show that if we impose additional requirements on construction blocks used in the described classes, we are able to prove that the resulting snark is critical or bicritical. Using appropriate snarks we construct infinite families of snarks with girth 6 and cyclic connectivity 5 or 6. Additionally, we construct all non-trivial snarks with girth 6, cyclic connectivity at most 5 and order 40.

**Keywords:** snark, irreducible, critical, bicritical, strictly critical, cyclical connectivity, girth, Tait colouring, flow

# Abstrakt

Snakry sú netriviálne bezmostové grafy, ktoré nie sú hranovo-3-zafarbiteľné. Táto práca rozširuje analýzu štruktúry malých bikritických snarkov od rovnakého autora na striktne kritické snarky. Táto analýza rozdeľuje skúmané snarky do niekoľkých nekonečných tried. Ukážeme, že pokiaľ položíme dodatočné podmienky na konštrukčné bloky použité v opísaných triedach, tak sme schopný dokázať, že výsledný snark je kritický, resp. bikritický. Vhodnou voľbou použitých snarkov skonštruujeme nekonečné triedy snarkov obvodu 6 a cyklickej súvislosti 5 alebo 6. Taktiež skonštruujeme všetky netriviálne snarky obvodu 6 s cyklickou súvislosťou najviac 5 a rádom 40.



**Kľúčové slová:** snark, ireducibilný, kritický, bikritický, striktne kritický, cyklický súvislosť, obvod, taitovo farbenie, tok

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# Introduction

Various famous conjectures in graph theory such as the cycle double cover conjecture or the 5-flow conjecture are sufficient to solve for cubic graphs. The Vizing's theorem says that every cubic graph is either 3-edge-colourable or 4-edge-colourable. It is not difficult to prove that the aforementioned conjectures are true for 3-edge colourable cubic graphs. Therefore, the class of cubic graphs that are not 3-edge colourable became the subject of comprehensive research since it could contain counterexamples to those conjectures. Such graphs are called *snarks* and they are additionally required to be bridgeless inasmuch as most of the involved conjectures are stated for bridgeless graphs.

Despite that snarks were rather rare in the early studies, at present, there is a plethora of infinite families of snarks. Also, in 2013 G. Brinkmann et al. generated all "non-trivial" snarks up to order 36 using a computer [4].

Although the essence of the definition of a snark consists of being a bridgeless cubic graph that is not 3-edge-colourable, we can observe that plenty of snarks are more or less trivial modifications of other snarks. R. Nedela and M. Škoviera have formalized some of those questions of the triviality of the snarks introducing *critical* and *bicritical* snarks [17]. As we shall discuss in Section 1.5, every non-critical snark can be constructed form some critical snark using a fairly simple operation.

In our previous work [19], we have analysed all bicritical cyclically 5-connected snarks up to order 36. Using these snarks, we were able to explain the uncolourability of every snark up to order 36. Also, we have introduced several infinite families covering those snarks.

In this work, we illustrate that if we lay down additional requirements for snarks used in our constructions, we are able to guarantee that the constructed snarks are bicritical. Moreover, we aim at snarks that are cyclically 5-connected and snarks with girth 6. As follows from the research of Chladný and Škoviera [6] and Carneiro [5], critical snarks that are not bicritical, or for short *strictly critical* snarks, also deserve an investigation. Therefore, we extend our analysis of small snarks to strictly critical snarks of order up to 36. Then we use structures observed in small snarks to construct infinite families of strictly critical snarks. Using appropriate snarks in these constructions, we shall be able to construct families of strictly critical snarks with girth 6 and even cyclic connectivity 6.

Furthermore, we shall make an attempt to construct other cyclically 6-connected snarks. We illustrate several methods of obtaining construction blocks with useful colouring properties. We also illustrate several techniques on how we can construct cyclically 6-connected snarks using described construction blocks. To offer suitable snarks for these constructions, we aim at generating snarks with girth 6 of order 40. We generate all such snarks with cyclic connectivity 4 or 5.

# Chapter 1

# Multipoles and snarks

## 1.1 Multipoles

Constructions of snarks are often described in terms of multipoles, that is, graphs allowed to contain dangling edges (see e. g. [9, 15, 17]). Formally, a *multipole* is a pair $M = (V(M), E(M))$, where $V(M)$ is a finite set of vertices and $E(M)$ is a finite set of edges. Every edge $e \in E(M)$ has two *ends*; an end of $e$ may or may not be incident with a vertex. The edges of a multipole $M$ are of four types.

1. A *link* is an edge whose ends are incident with two distinct vertices.

2. A *loop* is an edge whose ends are incident with the same vertex.

3. A *dangling edge* is an edge which has only one end incident with a vertex.

4. An *isolated edge* is an edge whose both ends are incident with no vertex.

A *semiedge* is an end of an edge that is incident with no vertex. The set of all semiedges of a multipole $M$ is denoted by $S(M)$. For the purpose of our work, we suppose that the set of semiedges $S(M)$ is endowed with a linear ordering of semiedges. If a multipole has $k$ semiedges, it is called a *$k$-pole* (for instance, a 0-pole is a graph). Note that every dangling edge contains one semiedge and every isolated edge has two semiedges. If there is only one dangling edge incident with a vertex $v$, we will denote the semiedge contained in the dangling edge by $(v)$.

Usually, it is convenient to divide the set $S(M)$ into pairwise disjoint sets $S_1, S_2, \ldots, S_n$, called *connectors*. Each connector is endowed with a linear ordering of semiedges. Then, the ordering of $S(M)$ is obtained naturally as the union of orderings for its connectors. A multipole $M$ with $n$ connectors $S_1, S_2, \ldots, S_n$ such that $|S_i| = c_i$ for $i \in \{1, 2, \ldots, n\}$ is denoted by $M(S_1, S_2, \ldots, S_n)$ and called a *$(c_1, c_2, \ldots, c_n)$-pole*. If a connector $S$ contains only one semiedge $s$, we will usually only write $s$ in place of $(s)$.

A *graph of a multipole* $M$ is the graph which is obtained from the multipole $M$ by removing all dangling edges and isolated edges; it is denoted by $G(M)$ and it is always subcubic. As one could expected, the *order* $|M|$ of a multipole $M$ is the number of its vertices and the *girth* $g(M)$ of a multipole $M$ is the length of the shortest cycle of $G(M)$. In this paper, we will only consider cubic multipoles, i. e. multipoles where each vertex is incident with three edge ends.

Further, we describe a method of joining two multipoles together. Let $e$ and $f$ be two edges (not necessary distinct) of a given multipole $M$ and let $e$, $f$ have semiedges $e'$, $f'$ respectively such that $e' \neq f'$. We can identify $e$ with $f$ and construct a new multipole $M'$ in the following way. If $e \neq f$, we replace $e$ and $f$ with a new edge $g$ whose ends are the other ends of $e$ and $f$, so $E(M') = (E(M) - \{e, f\}) \cup \{g\}$. If $e = f$, then $e$ is an isolated edge, and we simply put $E(M') = E(M) - \{e\}$ (the loop with no vertices is deleted since it has no impact on colourability). We say that the multipole $M'$ arises from $M$ by the *junction* of $e'$ and $f'$.

Let $M = M(e_1, e_2, \ldots, e_k)$ and $N = N(f_1, f_2, \ldots, f_k)$ be two $k$-poles. Then the *junction* $M * N$ of $M$ and $N$, is the graph that arises from the disjoint union $M \cup N$ by performing the junctions $e_i$ with $f_i$ for $i \in \{1, 2, \ldots, k\}$. Similarly, for connectors $S_1$, $S_2$ of the same size $k$ of a multipole $M$, we define the *junction* of the connectors $S_1$, $S_2$ as an operation consisting of $k$ individual junctions of $i$-th semiedge from $S_1$ with $i$-th semiedge from $S_2$ for $i \in \{1, 2, \ldots, k\}$.

## 1.2 Common constructions of multipoles

Throughout our work, we shall very often use several constructions of multipoles from graphs or other multipoles. Since the organisation of semiedges into connectors is sometimes ambiguous, we clarify it here. The only space for ambiguity is left in the order of semiedges in a connector as this order is not determined. However, in plenty of cases, the order of semiedges in a connector is not important.

Let $M(C_1, C_2, \ldots, C_k)$ be a multipole and $v$ one of its vertices. We can construct a new multipole $N(C_1, C_2, \ldots, C_k, C_{k+1})$ by removing the vertex $v$ from the multipole $M$ and putting the three semiedges formerly incident with $v$ in the connector $C_{k+1}$. Note that if the vertex $v$ is incident with a dangling edge $e$, the end of $e$ is retained in the multipole $N$ as an end of an isolated edge. We will denote the multipole $N$ by $M - v$. To keep our notation short, we shall write just $M - (v_1, v_2, \ldots, v_n)$ instead of $(((M - v_1) - v_2) - \ldots) - v_n$. Remark that this operation is not commutative; the order of the connectors changes with a change of the order of the removed vertices.

If we choose a link of a multipole $M(C_1, C_2, \ldots, C_k)$ and cut it into two dangling edges $e_1$, $e_2$ each incident with one end vertex of $e$, we construct a new multipole $N(C_1, C_2, \dots, C_k, (e_1, e_2))$ which we denote by $M - e$. Again, we shall write just $M - (e_1, e_2, \dots, e_n)$ instead of $(((M - e_1) - e_2) - \dots) - e_n$.

When we choose two adjacent vertices $u$, $v$ of a multipole $M(C_1, C_2, \dots, C_k)$, we can remove the vertices $u$, $v$ alongside one link between them. Denote the dangling edges formerly incident with $u$ by $e_1$, $e_2$ and the dangling edges formerly incident with $v$ by $f_1$, $f_2$. We denote the arisen multipole $N(C_1, C_2, \dots, C_k, (e_1, e_2), (f_1, f_2))$ by $M - [u, v]$. If there are two or more links between the vertices $u$, $v$, one of them is removed and the others remain in the $(2, 2)$-pole $M - [u, v]$ as isolated edges.

In the following example, we call attention to the difference of two similar notations that will appear pretty often as we would need to view a snark with two adjacent vertices removed sometimes as a $(3, 3)$-pole with an isolated edge and sometimes as a $(2, 2)$-pole.

**Example 1.** Let $u$, $v$ be an adjacent vertices of a snark $S$. The multipole $S - (u, v)$ is a $(3, 3)$-pole which contains in its first connector the semiedges formerly incident with $u$ and in its second connector the semiedges formerly incident with $v$. By the notation $M((e_1, e_2, e_3), (f_1, f_2, f_3)) = S - (u, v)$, we gave names to those semiedges (in some not specified order). For some $i$ and $j$, the semiedges $e_i$ and $f_j$ are ends of an isolated edge which corresponds to the former link between $u$, $v$.

On the other side, the multipole $S - [u, v]$ is a $(2, 2)$-pole $N((e_1, e_2), (f_1, f_2))$, where $e_1$, $e_2$ correspond to the edges distinct from the edge $uv$ incident with the vertex $u$ in the snark $S$ and similarly the semiedges $f_1$, $f_2$ correspond to the vertex $v$.

## 1.3 Tait colourings of multipoles

Generally, in our work, we shall colour multipoles instead of restricting to graphs, so also the dangling edges shall have assigned a colour. It is important to choose a convenient set of colours. As we shall show later, the set of non-zero elements of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has very good properties. We denote this set by $\mathbb{K}$.

**Definition 1.** Let $M$ be a multipole and let $\varphi : E(M) \to \mathbb{K}$ be a mapping assigning to each edge of $M$ a colour from $\mathbb{K}$. Then $\varphi$ naturally induces an assignment of colours to the edge ends of $M$. The mapping $\varphi$ is called a *3-edge-colouring* or simply a *colouring* of the multipole $M$, if for each vertex $v \in V(M)$ the three edge ends incident with $v$ have assigned pairwise distinct colours.

If there exists a colouring for a multipole $M$, we say that $M$ is *colourable*, otherwise *uncolourable*.

Using the colours from the set $\mathbb{K}$, we can use addition in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to analyse properties of the mapping $\varphi : E(M) \to \mathbb{K}$. If we denote $\delta(v)$ the set of edge ends incident with the vertex $v$, then obviously $\varphi$ is a colouring if and only if

$$\sum_{e\in\delta(v)} \varphi(e) = 0$$

for each vertex $v$. This equation is the Kirchhoff's law for flows in graphs. Thus, a colouring of a multipole induces a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow and vice-versa (for a definition of the flow see e. g. [8]). Considering that each element in $\mathbb{K}$ is its inverse, we do not have to distinguish the orientation of the flow. Such colouring using the colour set $\mathbb{K}$ is also called a *Tait colouring.*

When we view a colouring $\varphi$ of a multipole $M$ as a flow, then we can easily observe form the properties of a flow that

$$\sum_{e\in S(M)} \varphi(e) = 0.$$

Let $k_1$, $k_2$ and $k_3$ be the numbers of dangling edges coloured by colours $(0,1)$, $(1,0)$ and $(1,1)$, respectively. If the first entry of the sum of colours from $\mathbb{Z}_2 \times \mathbb{Z}_2$ has to be 0, then $k_1$ and $k_2$ have to have the same parity. The same holds for $k_2$ and $k_3$. This result is known as the Parity Lemma which was first published by Blanuša [2] and then by Descartes [7], originally stated for the numbers of the used colours in an edge-cut of a 3-edge-colourable graph.

**Theorem 1.1** (Parity Lemma)**.** *Let $M$ be a $k$-pole and let $k_1$, $k_2$ and $k_3$ be the numbers of dangling edges coloured by colour $(0,1)$, $(1,0)$ and $(1,1)$, respectively. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

## 1.4 Snarks and their triviality

Before we define snarks, we explain one crucial property we shall use in the characterization of snarks. A very important property of cubic graphs is connectivity, precisely the edge-connectivity. However, as each cubic graph is at most 3-edge-connected, there is a need to better distinguish connectivity of cubic graphs. We say that a cubic graph $G$ is *cyclically $k$-edge-connected* if there is no set $S$ containing less than $k$ edges such that the graph $G - S$ contains at least two components with a cycle. A *cyclic edge-connectivity* of a graph $G$ is the smallest number $k$ such that $G$ is cyclically $k$-edge-connected and it is denoted by $\lambda_c(G)$. If a graph $G$ has no cycle separating cut (e. g. $K_4$, $K_{3,3}$), then we put $\lambda_c(G) = \infty$. As the cyclic edge-connectivity and cyclic vertex-connectivity of a given cubic graph are equal [16], we shall omit the word edge and say only cyclically $k$-connected and cyclic connectivity.

In our work, we will use a definition of a snark which follows several famous conjectures which are proposed for bridgeless graphs.

**Definition 2.** A *snark* is a connected bridgeless cubic graph which is not 3-edge-colourable.

As snarks often serve as counterexamples, in many definitions of snarks occurs in some form a word non-trivial. This follows that many snarks are only small modifications of other snarks.

If a snark $S$ contains a triangle, we can replace it by a single vertex as shown in Figure 1.1 resulting in the graph $S'$. It can be easily shown that the graph $S$ is colourable if and only if $S'$ is colourable. We can look at this from the other side as an operation allowing us to construct infinitely many snarks from a given one, but they will be only trivial modifications of the former one.

Consider a snark $S$ with a cycle of length 4. We can replace it by two parallel edges as in Figure 1.1 resulting in the graph $S'$. Again, it is easy to see that if $S'$ is colourable, then $S$ is also colourable. Note that this does not in general work in a reverse way.

Figure 1.1: Removing triangles and 4-cycles in a snark

Now we take a look at snarks with small cyclic connectivity. From the Parity Lemma, it is easy to see that a cubic graph with a bridge is uncolourable.

Consider a snark $S$ with an edge-cut of size 2 which decompose it into two 2-poles $M$, $N$. If both $M$ and $N$ are colourable, then by the Parity Lemma, both dangling edges in $M$ and also in $N$ have the same colour, so we can extend the colourings of $M$ and $N$ to a colouring of the snark $S$. Therefore at least one of the components $M$, $N$ is uncolourable. So we can construct a smaller snark from the snark $S$ by joining the two semiedges in the uncolourable component.

When a snark $S$ has a 3-edge-cut, then again, one of the components $M$, $N$ has to be uncolourable. Otherwise, the dangling edges of $M$ and $N$ would have three different colours by the Parity Lemma. Thus, by connecting the dangling edges of the uncolourable component of $S$, we can construct a smaller snark.

These were the most common properties of non-trivial snarks which also appear directly in many definitions of snarks. To distinguish these properties we will call snarks trivial and non-trivial.

**Definition 3.** A snark with girth at least 5 and cyclic connectivity at least 4 is called *non-trivial*. The other snarks are called *trivial*.

Often we shall work with a class of snarks with given cyclic connectivity, girth and sometimes also order. To make our writings more compendious, we denote the class of all snarks with cyclic connectivity at least $c$ and girth at least $g$ by $\mathcal{S}(c, g)$. Moreover, the subclass of $\mathcal{S}(c, g)$ consisting of snarks of order exactly $n$ is denoted by $\mathcal{S}(c, g, n)$. Sometimes, we would like to consider snarks of the class $\mathcal{S}(c, g)$ with cyclic connectivity exactly $c$, we denote such class by $\mathcal{S}(= c, g)$. Analogously, the denotation $\mathcal{S}(\leq c, g)$ stands for a class of snarks $S$ with $\lambda_c(S) \leq c$ and $g(S) \geq g$. We also use a similar denotation for the girth: $\mathcal{S}(c, = g) = \mathcal{S}(c, g) - \mathcal{S}(c, g+1)$, $\mathcal{S}(c, \leq g) = \mathcal{S}(c, 1) - \mathcal{S}(c, g+1)$ and order: $\mathcal{S}(c, g, \leq n) = \cup_{m=1}^{n} \mathcal{S}(c, g, m)$, $\mathcal{S}(c, g, \geq n) = \cup_{m=n}^{\infty} \mathcal{S}(c, g, m)$.

## 1.5 Critical and bicritical snarks

With the discussion about the non-triviality of snarks, we can go further. One such approach consists in asking how many and which vertices of a snark we can remove to get a colourable graph. Because if we can remove many vertices from a snark, it contains in some way redundant parts.

By the Parity Lemma, removing one vertex from a snark leaves always an uncolourable graph, so we have to remove at least two vertices. A pair of vertices $\{u, v\}$ of a snark $S$ is called *non-removable*, if the 4-pole $S - [u, v]$ is colourable, otherwise it is called *removable*. A snark $S$ is called *critical* if every pair of distinct adjacent vertices in $S$ is non-removable. Furthermore, if every pair of distinct vertices in $S$ is non-removable, we call the snark $S$ *bicritical*. These notions were introduced by Nedela and Škoviera in [17].

A similar approach can be done with edges of a snark. Again, if we split only one edge in a snark, we get an uncolourable 2-pole. A pair of edges $\{e, f\}$ of a snark $S$ is called *non-removable* if the $(2, 2)$-pole $S - (e, f)$ is colourable, otherwise it is called *removable*. Chladný and Škoviera introduced also a stronger concept than the non-removable pair of edges.

**Definition 4** (Chladný and Škoviera [6])**.** A pair of distinct edges $\{e, f\}$ of a snark $G$ is *essential* if it is non-removable and for every 2-valent vertex $v$ of the graph $G - \{e, f\}$, the graph obtained from $G - \{e, f\}$ by suppressing $v$ is colourable.

Every pair of essential edges $\{e, f\}$ is non-removable, so the 4-pole $M - (e, f)$ is colourable. Moreover, this notion requires that other four 3-poles $M_1$, $M_2$, $M_3$, $M_4$ illustrated in Figure 1.2 are colourable.

Essential pairs of edges play a crucial role in the study of snarks of cyclic connectivity 4. We shall also use this notion to obtain specific colourings of various multipoles.

Nedela and Škoviera also introduced another concept of measuring a triviality of snarks. Firstly, they generalized the decomposition theorems of Goldberg [10] and

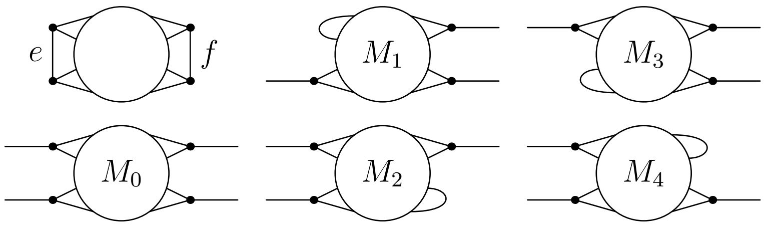


Figure 1.2: Five multipoles that are required to be colourable for a pair of the edges $\{e, f\}$ to be essential

Cameron et al. [18] and published the following theorem [17].

**Theorem 1.2.** *Let $G$ be a snark and let $k \geq 1$ be an integer. Then there exists an integer function $\kappa(k)$ such that if $G = M * N$ is a $k$-junction of two $k$-poles, then one of the following statements holds.*

*(a) One of $M$ and $N$ is not colourable.*

*(b) Both $M$ and $N$ can be extended to snarks $\bar{M}$ and $\bar{N}$ by applying the junction with $k$-poles $M'$ and $N'$, each having at most $\kappa(k)$ vertices. Moreover, $|\bar{M}| \leq |G|$ and $|\bar{N}| \leq |G|$.*

Theorem 1.2 describes two cases which we can observe in edge-cuts of snarks. We take a closer look at the part (a). Let us consider a snark $G = M * N$ satisfying the condition (a), so let us say that $M$ is uncolourable. Then $M$ can be extended to a snark $\bar{M} \supseteq M$ of order not greater than $|G|$ by simply restoring its 3-regularity. In this case, we have reduced snark $G$ to the snark $\bar{M}$ which is called a *$k$-reduction* of $G$. If additionally $|\bar{M}| < |G|$, then we call such $k$-reduction *proper*.

If the snark $G$ has any proper $k$-reduction, the essence of its uncolourability can be found in the smaller snark $\bar{M}$ and the $k$-pole $N$ contained in $G$ can be viewed as redundant. Therefore, snarks which do not admit $k$-reductions can be considered as "sufficiently non-trivial". To formalize this approach, we define that a snark is called *$k$-irreducible* if it has no proper $m$-reduction for each $m < k$. If a snark is $k$-irreducible for each $k$, then it is called *irreducible*. The following characterisation of $k$-irreducible snarks has been given by Nedela and Škoviera [17].

**Theorem 1.3.** *Let $G$ be a snark. Then the following statements hold true.*

*(a) If $1 \leq k \leq 4$, then $G$ is $k$-irreducible if and only if it is cyclically $k$-connected.*

*(b) If $5 \leq k \leq 6$, then $G$ is $k$-irreducible if and only if it is critical.*

*(c) If $k \geq 7$, then $G$ is $k$-irreducible if and only if it is bicritical.*

**Corollary 1.1.** *A snark is irreducible if and only if it is bicritical.*

If we have a snark $S$ that is not critical, i. e. the 4-pole $S-[u,v]$ is uncolourable for some adjacent vertices $u,v \in V(S)$, then we can complete the 4-pole $S-[u,v]$ to a snark $T$ by performing junctions of its two pairs of semiedges. The snark $T$ has a smaller order than the snark $S$. In reverse, we can choose two distinct edges $e$, $f$ of a snark $T$. Then we subdivide each of $f$ and $g$ with a new vertex and add an edge between the two new added vertices. We denote the resulted graph by $S$ and we say that $S$ was obtained by an I-*extension* across $e$ and $f$. Nedela and Škoviera proved in [17] that the graph $S$ is a snark if and only if the pair of the edges $\{e,f\}$ is removable in $S$.

This means that we can construct any non-critical snark from a smaller snark by an $I$-extension. Therefore, research of critical snarks is needed. For instance, if we would like to generate all non-trivial snarks of order 38, it is sufficient to generate critical ones since the remaining snarks can be obtained using I-extensions.

## 1.6 Colouring sets

To study the colourability of snarks, we will decompose a snark into smaller multipoles and look at possible colourings of their semiedges. Because we will work mostly with critical snarks, each multipole would be colourable. One such multipole has several possibilities how its dangling edges can be coloured. However, if we look at all multipoles contained in a snark, we would not find any common colouring which would assign to all corresponding dangling edges the same colours.

**Definition 5.** Let $M(e_1,e_2,\dots,e_k)$ be a $k$-pole. The *colouring set* of the $k$-pole $M$ is the set

$$\mathrm{Col}(M) = \{(\varphi(e_1),\varphi(e_2),\dots,\varphi(e_k)) \mid \varphi \text{ is a Tait colouring of } M\}.$$

The $k$-tuple $(\varphi(e_1),\varphi(e_2),\dots,\varphi(e_k))$ is denoted by $\varphi(M)$ for a given colouring $\varphi$ of the $k$-pole $M$.

**Definition 6.** Let $S=(e_1,e_2,\dots,e_k)$ be a connector of a multipole $M$. The *flow through* $S$ it the value $\varphi_\Sigma(S)=\sum_{i=1}^{k}\varphi(e_i)$. The $k$-tuple $(\varphi(e_1),\varphi(e_2),\dots,\varphi(e_k))$ is denoted by $\varphi(S)$.

**Definition 7.** A connector $S$ is called *proper* if $\varphi_*(S)\neq 0$ for each colouring $\varphi$ of the multipole $M$. A multipole is called *proper* if all of its connectors are proper.

## 1.7 Substitutions

In section 1.4, we introduced two simple operations allowing us to construct a snark from another one. In one, we replaced a vertex of a snark with a triangle, in other, we replaced a 4-cycle with two parallel edges (see also Figure 1.1). In general, we can replace a $k$-pole $M_1$ contained in a graph $G_1 = M_1 * R$ with another $k$-pole $M_2$ to obtain a graph $G_2 = M_2 * R$. For some pairs of $k$-poles $M_1$, $M_2$, the graph $G_1$ is uncolourable if and only if $G_2$ is uncolourable (e. g. a vertex and a triangle), for some other pairs, the uncolourability of $G_1$ implies the uncolourability of $G_2$ and not vice versa (a 4-cycle and two parallel edges). Using the relations between the colouring sets of $M_1$, $M_2$, we are able to describe relations between the colourability of the graphs $G_1$, $G_2$ [9].

A $k$-pole $M_1$ is said to be *colour-contained* in a $k$-pole $M_2$ if $\mathrm{Col}(M_1) \subseteq \mathrm{Col}(M_2)$. If $\mathrm{Col}(M_1) = \mathrm{Col}(M_2)$, then we say that the multipoles $M_1$ and $M_2$ are *colour-equivalent*. The following result was published by Fiol [9].

**Lemma 1.1.** *If a $k$-pole $M_1$ is colour-contained in a $k$-pole $M_2$ and $M_2 * R$ is uncolourable for some $k$-pole $R$, then the graph $M_1 * R$ is also uncolourable. Moreover, if $M_1$ and $M_2$ are colour-equivalent $k$-poles, then the graph $M_1 * R$ is uncolourable if and only if the graph $M_2 * R$ is uncolourable.*

A similar result can also be applied to a substitution of a multipole which is contained in a larger multipole.

**Lemma 1.2.** *Let $M_1$ and $M_2$ be two colour-equivalent $k$-poles such that $M_1$ is contained in some multipole $M$. Denote the multipole obtained from $M$ by substituting the $k$-pole $M_1$ for the $k$-pole $M_2$. Then* $\mathrm{Col}(M) = \mathrm{Col}(N)$.

# Chapter 2

# Commonly used multipoles

In many snarks, we can observe several similarities. There are some types of multipoles which occur in many snarks. They are constructed from smaller snarks by removing some vertices or splitting some edges. Those multipoles are commonly used in various constructions of infinite families of snarks [14], [15]. In this chapter, we describe the multipoles we shall use in our work and their properties.

## 2.1 Paths and cycles

At first, we describe some common simple multipoles. Although denotation for graphs which corresponds to them is already known, we need to specify connectors of those multipoles.

**Definition 8.** For $k \geq 1$, a *path of length* $k$ is a $(2, 2, k-1)$-pole

$$P_k((i_1, i_2), (o_1, o_2), (r_1, r_2, \ldots, r_{k-1}))$$

whose graph is a path $v_0 v_1 \ldots v_k$, the dangling edges $i_1$, $i_2$ are incident with the vertex $v_0$, the dangling edges $o_1$, $o_2$ are incident with the vertex $v_k$ and the dangling edge $r_i$ is incident with $v_i$ for $1 \leq i < k$.

**Definition 9.** For $k \geq 1$, a *cycle of length* $k$, or for short a $k$*-cycle*, is a $k$-pole $C_k(e_1, e_2, \ldots, e_k)$ consisting of a cycle $v_1 v_2 \ldots v_k$ where the dangling edge $e_i$ is incident the vertex $v$ for each $1 \leq i \leq k$.

At some times, a 6-cycle is better represented as a $(2, 2, 2)$-pole. Then we shall use the following notation

**Definition 10.** A $(2, 2, 2)$-pole consisting of a cycle of length 6 $v_1 v_2 v_3 v_4 v_5 v_6$ with a dangling edge $e_i$ incident with the vertex $v_i$ for each $i \in \{1, 2, 3, 4, 5, 6\}$ is denoted by $C'_6((e_1, e_3), (e_2, e_4), (e_3, e_6))$.

## 2.2 Isochromatic

*Isochromatic* is a $(2,2)$-pole of the form $S-[u,v]$ for a snark $S$ and a pair of its adjacent vertices $u$, $v$. Since we work mostly with cyclically 5-connected snarks, it shall not appear in our constructions, however, we shall often use its colouring properties to obtain colourings of multipoles constructed from critical snarks. The following lemma follows from the Goldberg decomposition theorem [10].

**Lemma 2.1.** *Let $S$ be a snark and $\{u,v\}$ a pair of its non-removable vertices. Then*

$$\mathrm{Col}(S-[u,v]) = \{(a,a,b,b) \mid a,b \in \mathbb{K}\}.$$

*Proof.* Since the pair of the vertices $\{u,v\}$ is non-removable, the $(2,2)$-pole $M((e_1,e_2),(e_3,e_4)) = S-[u,v]$ is colourable by a colouring $\varphi$. From the Parity lemma, we know that $\varphi(e_1)=\varphi(e_2)=a$ and $\varphi(e_3)=\varphi(e_4)=b$ for some $a,b\in\mathbb{K}$. For any $a\neq c\in\mathbb{K}$, we can obtain the colours $c$ on the semiedges $e_1$, $e_2$ by interchanging the colours on the $a$–$c$ Kempe chain. This Kempe chain has to have its ends in the same connector, otherwise after interchanging the colours, we could extend the new colouring of $M$ to a colouring of the snark $S$. □

## 2.3 Negator

Let $S$ be a snark and $uwv$ a path of length two in $S$. A *negator* is a $(2,2,1)$-pole $N(I,O,r) = S-uwv$ which is denoted by $\mathrm{Neg}(S,u,v)$. In other words, the negator $N$ was constructed from the snark $S$ by removing the path $uwv$ while the connector $I$ contains its semiedges formerly incident with $u$, the connector $O$ contains the semiedges formerly incident with $v$ and the semiedge $r$ is the remaining semiedge formerly incident with $w$. This semiedge $r$ is also called a *residual semiedge*.

We have to say that the notation $\mathrm{Neg}(S,u,v)$ is ambiguous if there is more than one common neighbour of the vertices $u$, $v$. This can happen only if the girth of the snark $S$ is at most 4, so the snark $S$ is trivial. As we study primarily non-trivial snarks, we will use this notation (used also in [15]) and the possible ambiguity will play no significant role in our work.

For each colouring of a negator $N(I,O,r) = \mathrm{Neg}(S,u,v)$, the flow through exactly one of its connectors $I$, $O$ is zero. Otherwise, we could extend such colouring of

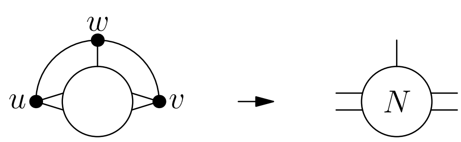


Figure 2.1: The snark $S$ and a symbolic representation of the negator $N = \mathrm{Neg}(S,u,v)$

$N$ to a colouring of the snark $S$ [15]. By the Parity Lemma, the flow through the other connector is the same as through the residual semiedge. In other words, the colouring set of every negator is a subset of

$$C = \{(x, x, a, b, a + b) \mid x, a, b \in \mathbb{K},\ a \neq b\} \cup \{(a, b, x, x, a + b) \mid x, a, b \in \mathbb{K},\ a \neq b\}.$$

A negator whose colouring set is equal to $C$ is called *perfect*. The characterisation of perfect negators was published by Máčajová and Škoviera in [15].

**Theorem 2.1.** *A negator $N = S - uwv$ is perfect if and only if each of the pairs $\{u, w\}$ and $\{v, w\}$ of adjacent vertices is non-removable in $S$.*

Observe that each negator constructed from a critical snark is perfect.

## 2.4 Proper (2,3)-pole

Let $S$ be a snark, $v$ a vertex in $S$ and $e$ an edge in $S$ not incident with $v$. The $(2, 3)$-pole $T(D, E) = (S - e) - v$ is called a *proper* $(2, 3)$*-pole*.

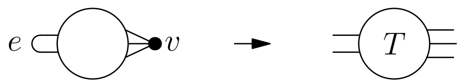


Figure 2.2: The snark $S$ and a symbolic representation of a proper $(2, 3)$-pole $T$

As its name says, the proper $(2, 3)$-pole is always proper, i. e. flow through both of its connectors is always non-zero [19].

We call a proper $(2, 3)$-pole $T$ *perfect* if

$$\mathrm{Col}(T) = \{(a, b, c, d, e) \in \mathbb{K}^5 \mid a + b + c + d + e = 0,\ a + b \neq 0 \neq c + d + e\}.$$

## 2.5 Odd (2,2,2)-pole

Let $G$ be a snark and $v$ one its vertex with neighbours $u_1$, $u_2$ and $u_3$. Remove from $G$ the vertices $v$, $u_1$, $u_2$, $u_3$. For $i \in \{1, 2, 3\}$, denote the two semiedges incident with $u_i$ by $e_i$ and $f_i$. Group these semiedges into connectors $S_i = (e_i, f_i)$. The resulting $(2, 2, 2)$-pole $H(S_1, S_2, S_3)$ is called an *odd* $(2, 2, 2)$*-pole*.

The name of this multipole is derived from its colouring properties and a similar name was used by Goldberg [10]. For each colouring $\varphi$ of $H$, the number of connectors of $H$ having a zero flow is odd. As the Parity Lemma says, it is impossible that flow through exactly two connectors is zero, thus, it is sufficient to show that flow through at least one of the connectors $S_1$, $S_2$, $S_3$ is zero.

Suppose the contrary. If $\varphi_*(S_i) = a_i \neq 0$ for every $i \in \{1, 2, 3\}$, then we can connect the semiedges $e_i$ and $f_i$ with a new vertex $u_i$. The third semiedge incident with $u_i$ can be coloured by the colour $a_i$. From the Parity lemma, we have that $a_1 + a_2 + a_3 = 0$, so all three new semiedges incident with $u_1$, $u_2$ and $u_3$, respectively have pairwise different colours, so we can connect them to a new vertex $v$ resulting in the snark $G$ witch is colourable by a extension of $\varphi$ and that is a contradiction.

A simple example of an odd $(2, 2, 2)$-pole is the 6-cycle $C_6'$. It arises from the Petersen graph by removing an arbitrary vertex with its three neighbours.

We shall denote the $(2, 2, 2)$-pole removed from the snark $G$ as $V_4(S_1, S_2, S_3)$. It consist of four vertices $v$, $u_1$, $u_2$ and $u_3$, where $v$ is the common neighbour of $u_1$, $u_2$, $u_3$. The connector $S_i$ contains the two semiedges incident with $u_i$ for $i \in \{1, 2, 3\}$. The colouring set of $V_4$ is the set

$$\mathrm{Col}(V_4) = \{(a, b, c, d, e, f) \in \mathbb{K} \mid a \neq b,\ c \neq d,\ e \neq f,\ a + b + c + d + e + f = 0\}.$$

## 2.6 Multipoles from Isaacs snarks

The family of Isaacs snarks was the first known infinite family of snarks [11]. The Isaacs snark $J_k$ consists of $k$ copies of the $(3, 3)$-pole $Y((i_1, i_2, i_3), (o_1, o_2, o_3))$ depicted in Figure 2.3 connected in a circle for an odd integer $k \geq 3$. The examples of the snarks $J_5$ and $J_7$ are shown in Figure 2.4. A $(3, 3)$-pole consisting of $n$ copies $(Y_i(I_i, O_i))_{i=1}^n$ of the $(3, 3)$-pole $Y$ where junctions of the connectors $O_i$ and $I_{i+1}$ for $1 \leq i < n$ were performed is denoted by $Y_n$.

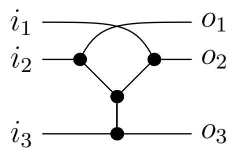


Figure 2.3: The $(3, 3)$-pole $Y$ used in the construction of Isaacs snarks

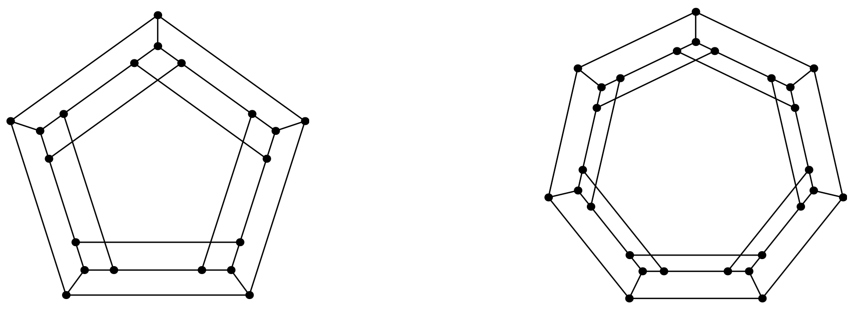

Figure 2.4: The Isaacs snarks $J_5$ and $J_7$

Isaacs snarks have very satisfying properties, thus we shall use them a lot in our constructions. For an odd $n \geq 5$, all Isaacs snarks are irreducible [17]. Moreover, each pair of their edges is essential [6]. Along with the Petersen graph and the double star snark, the Isaacs snarks are the only known snarks with this property [6].

Another useful property of the Isaacs snarks is their inductive construction. Therefore, additional properties of Isaacs snarks can be proven using mathematical induction. In such proofs, we shall often use the following lemma which can be found also in [17, Propostion 4.7].

**Lemma 2.2.** *For each $n \geq 1$, one has that* $\mathrm{Col}(Y_{2n}) = \mathrm{Col}(Y_2)$.

Since Isaacs snarks except $J_3$ are irreducible, every negator constructed from them is perfect. The same holds also for the proper $(2,3)$-poles as we prove in the following lemma.

**Lemma 2.3.** *Let $T$ be a proper $(2,3)$-pole $(J_n - e) - v$ for odd $n \geq 5$ and non-incident edge $e$ and vertex $v$. Then the proper $(2,3)$-pole $T$ is perfect.*

*Proof.* Using a computer, we verified that the lemma holds for $J_5$ and $J_7$. Suppose that the lemma holds for some odd $n \geq 7$. Then the proper $(2,3)$-pole $T = (J_{n+2} - e) - v$ contains the 6-pole $Y_4$ which is colour-equivalent to the 6-pole $Y_2$. Therefore, the colouring set of $T$ is equal to the colouring set of some proper $(2,3)$-pole constructed from $J_n$ which is perfect by the induction hypothesis. □

**Lemma 2.4.** *Let $u, v$ be arbitrary non-adjacent vertices of the Isaacs snark $J_n$ for some odd $n \geq 5$. Then the $(3,3)$-pole $M((e_1, e_2, e_3), (f_1, f_2, f_3)) = J_n - (u, v)$ is colourable by a colouring $\varphi$ such that $\varphi(e_1) = a$, $\varphi(e_2) + \varphi(e_3) = b$, $\varphi(f_1) = c$ and $\varphi(f_2) = \varphi(f_3) = d$ for any $\{a, b, c\} = \mathbb{K}$ and any $d \in K$.*

*Proof.* Since by Lemma 2.3, the proper $(2,3)$-pole $T((g_1, g_2), (f_1, f_2, f_3)) = (J_n - e_1) - v$ is perfect, it is colourable by a colouring $\varphi$ such that $\varphi(g_1) = a$, $\varphi(g_2) = b$, $\varphi(f_1) = c$ and $\varphi(f_2) = \varphi(f_3) = d$. The $(3,3)$-pole $M$ is contained in the $(2,3)$-pole $T$ where the semiedge $e_1$ corresponds to the semiedge $g_1$ of $T$ and the semiedges $e_2$, $e_3$ correspond to the two edge ends in $T$ which are adjacent to the semiedge $g_2$ and thus $\varphi(e_2) + \varphi(e_3) = \varphi(g_2) = b$. Hence the colouring $\varphi$ has the desired properties. □

## 2.7 NT (2,3)-pole

Suppose that we have a negator $N$ and a proper $(2,3)$-pole $T$ and we connect them as depicted in Figure 2.5. The constructed $(2,3)$-pole is denoted by $\mathrm{NT}(N, T)$. If we denote the $(2,3)$-pole which contains one isolated edge with ends $e_1$, $e_2$ and one vertex incident with three dangling edges $f_1$, $f_2$, $f_3$ by $M_{ev}((e_1, e_2), (f_1, f_2, f_3))$, then $\mathrm{Col}(\mathrm{NT}(N, T)) \subseteq \mathrm{Col}(M_{ev})$. Moreover, if both the negator $N$ and the proper $(2,3)$-pole $T$ are perfect, then the $(2,3)$-poles $\mathrm{NT}(N, T)$ and $M_{ev}$ are colour-equivalent. Both

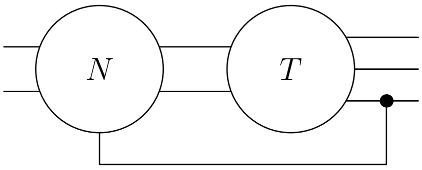


Figure 2.5

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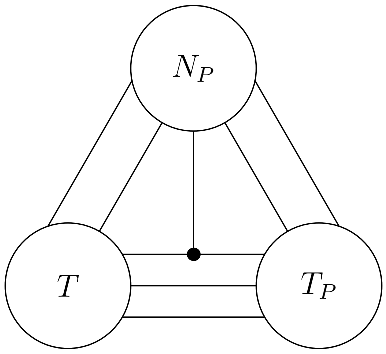



Figure 2.6

these results was proven in [19, Section 3.4]. Therefore, if we substitute an isomorphic copy of $M_{ev}$ in some snark $S$ for a $(2,3)$-pole $\mathrm{NT}(N,T)$, then we always get a snark $S'$. Note that the snark $S' = T' * \mathrm{NT}(N,T)$ has the structure depicted in 2.6 where $T' = (S-e)-v$, so $T'$ is a proper $(2,3)$-pole .

A $(2,3)$-pole $\mathrm{NT}(N_P,T_P)$ consisting of a negator $N_P$ from the Petersen graph and a proper $(2,3)$-pole $T_P$ from the Petersen graph is denoted by $P_{\mathrm{NT}}$.

# Chapter 3

# Strictly critical snarks

Almost all critical snarks up to order 36 are also bicritical. Chladný and Škoviera studied criticality and bicriticality of cyclically 4-connected snarks and stated that critical snarks that are not bicritical deserve a research. These snarks are called *strictly critical* and they were studied also in [5]. The smallest strictly critical snark has order 32. Chladný and Škoviera showed in [6] that there exists a strictly critical snark of each even order greater than 30. Since we understand strictly critical snarks with cyclic connectivity 4 pretty well [6], we aim at cyclically 5-connected ones. Using the methods developed in our bachelor thesis [19], we analyse strictly critical snarks up to order 36. Then, we generalize some of them to infinite classes of strictly critical snarks containing snarks with strong properties—girth 6 and even the cyclic connectivity 6 (Theorem 3.3) which solves Problem 6.3 in [6].

## 3.1 Structure of strictly critical snarks up to order 36

Among the snarks of order at most 36, there are only 84 cyclically 5-connected strictly critical snarks, all having 36 vertices. Of those, 77 arose by a substitution of a copy of the $(2,3)$-pole $M_{ev}$ by the $(2,3)$-pole $P_{NT}$ (see Section 2.7) in some non-critical snark of order 20. The structure of the remaining 7 snarks is very similar, they arise from the Petersen graph by substituting a copy of $V_4$ with the multipole $M$ described below. Recall that $V_4$ consists of one vertex with its three neighbours.

Let $T_1$, $T_2$, $T_3$ be three perfect proper $(2,3)$-poles. We connect them with three additional vertices into a $(2,2,2)$-pole $M$ as shown in Fig. 3.1 which we denote by $\mathrm{TTT}_{sc}(T_1, T_2, T_3)$. The $(2,2,2)$-pole $M$ is obviously proper, so $\mathrm{Col}(\mathrm{TTT}_{sc}(T_1, T_2, T_3)) \subseteq \mathrm{Col}(V_4)$. Note that the three vertices of the multipole $M$ contained in none of the proper $(2,3)$-poles can be removed, since the three connectors of $M$ remains still proper. Hence any snark containing $M$ is not bicritical. All 7 snarks contain proper $(2,3)$-poles taken from the Petersen graph.

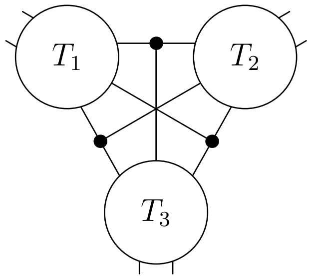



Figure 3.1: A $(2, 2, 2)$-pole contained in 7 strictly critical snarks from $\mathcal{S}(5, 5, 36)$

## 3.2 Strictly critical snarks from S(5, 6)

In this section, we generalize the structure of the 7 strictly critical snarks of order 36 and describe an infinite class of strictly critical snarks from $\mathcal{S}(5, 6)$. For the simplicity, we shall always replace $V_4$ in the Petersen graph. In other words, the resulting snark will be of the form $(2, 2, 2)$-pole $C_6' * \mathrm{TTT}_{sc}(T_1, T_2, T_3)$ for some proper $(2, 3)$-poles $T_1$, $T_2$, $T_3$. Recall that $C_6'((e_1, e_3), (e_2, e_4), (e_3, e_4))$ is the denotation of the 6-cycle $v_1v_2v_3v_4v_5v_6$ organized as a $(2, 2, 2)$-pole where the dangling edge $e_i$ is incident with the vertex $v_i$.

Since a proper $(2, 3)$-pole can be uncolourable, even if it is constructed from a critical snark, it is not sufficient to take the proper $(2, 3)$-poles $T_1$, $T_2$, $T_3$ from critical snarks. To be able to prove that the resulting snark is critical, we require from the used proper $(2, 3)$-poles an additional, rather technical, property.

**Definition 11.** A proper $(2, 3)$-pole $T = (S - e) - v$ is called *good* if for every end vertex $w$ of the link $e$ and every edges $f$, $g$ such that $f$ is incident with $w$ and $g$ is incident with $v$, the pair of edges $\{f, g\}$ is essential in $S$.

**Lemma 3.1.** *Let $S$ be a snark and $u$, $v$ two its vertices such that for each edges $e$, $f$ incident with $u$, $v$, respectively, the pair of edges $\{e, f\}$ is essential in $S$. Then there exists a colouring $\varphi$ of the 6-pole $M((e_1, e_2, e_3), (f_1, f_2, f_3)) = S - (u, v)$ such that $\varphi(e_1) = a$, $\varphi(e_2) = \varphi(e_3)$, $\varphi(f_1) = b$ and $\varphi(f_2) + \varphi(f_3) = c$ for $\{a, b, c\} = \mathbb{K}$.*

*Proof.* Since the pair of the edges $\{e_1, f_1\}$ is essential in $S$, the 3-pole $(S - (e_1, f_1)) \sim u$ is colourable and using this colouring, the desired colouring of $S - (u, v)$ can be obtained in a straightforward way. □

**Theorem 3.1.** *Let $T_1$, $T_2$ and $T_3$ be three perfect good proper $(2, 3)$-poles. Then the snark $C_6' * \mathrm{TTT}_{sc}(T_1, T_2, T_3)$ is strictly critical.*

*Proof.* We shall suppose that $T_i = (S_i - e_i) - v_i$ for each $i \in \{1, 2, 3\}$. Clearly, the snark $S$ is not bicritical, so it is sufficient to show that it is critical, precisely, that for arbitrary adjacent vertices $x$, $y$ of $S$, the 4-pole $M = S - [x, y]$ is colourable. We shall distinguish four cases.

**Case (i).** When both the vertices $x$, $y$ belong to the 6-cycle the 4-pole $M$ can be coloured as shown in Figure 3.2. All proper $(2,3)$-poles admit colourings as in the figure since they are perfect.

**Case (ii).** The vertex $x$ is from the 6-cycle and the vertex $y$ is from some proper $(2,3)$-pole, say, $T_1$. The colouring of the 4-pole $M$ is depicted in Figure 3.3. By Lemma 3.1, the 6-pole $T_1 - y$ admits such colouring as it is equal to $S - (y, v_1)$ and $T$ is a good proper $(2,3)$-pole.

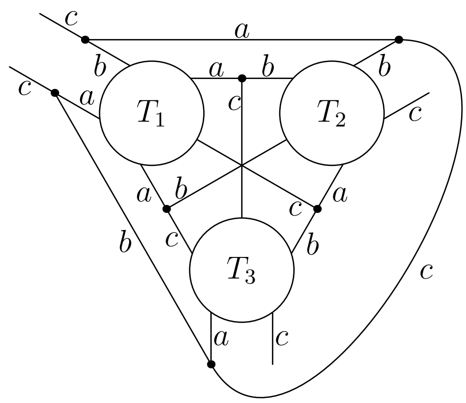



Figure 3.2: Colouring for case (i)

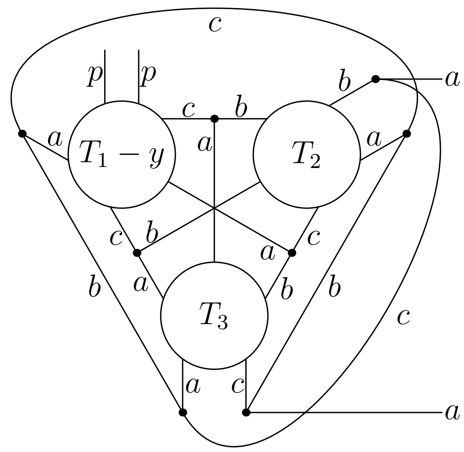



Figure 3.3: Colouring for case (ii)

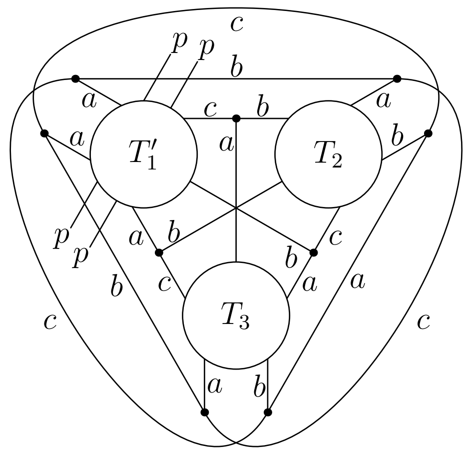
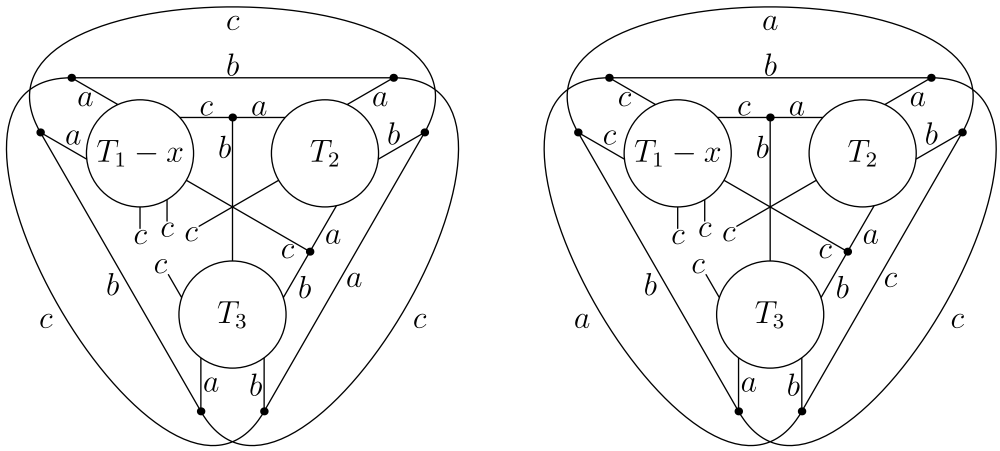



Figure 3.4: Colouring for case (iii)

**Case (iii).** Both the vertices $x$, $y$ belong to the same proper $(2,3)$-pole, say, $T_1$. The sought colouring of $S - [x,y]$ is illustrated in Figure 3.4 while the colouring of the 9-pole $T_1' = T_1 - (x,y)$ can be obtained in the following way. We colour all the dangling edges of the 4-pole $S_1 - [x,y]$ by the same colour $p$ (see Lemma 2.1). Let the link $e_1$ of $S_1 - [x,y]$ is coloured by the colour $a \in \mathbb{K}$. The links incident with the vertex $v_1$ have pairwise distinct colours $a$, $b$, $c$ in some order, which is not important due to the symmetry of the snark $S$. After the removal of the vertex $v_1$ and splitting the link $e_1$, we get the sought colouring of the 9-pole $T_1 - (x, y)$.

**Case (iv).** The vertex $x$ is from some proper $(2,3)$-pole, say, $T_1$ and $y$ is one of the vertices connecting two proper $(2,3)$-poles, say $T_1$ and $T_3$. Initially, we find a colouring of the 6-pole $T_1 - x \cong (S_1 - [v_1, x]) - e_1$. Since the snark $S_1$ is critical, the 4-pole $S_1 - [v_1, x]$ is colourable in such a way that all its dangling edges are coloured by the same colour $c$. Denote the colour of the link $e_1$ of $S_1 - [v_1, x]$ by $p$. After splitting the link $e_1$, we obtain the 6-pole $T_1 - y \cong (S_1 - [v_1, x]) - e_1$ with its semiedges coloured by $(c, c, c, c, p, p)$. If $p \neq c$, the colouring of the 4-pole $M = S - [x, y]$ is depicted in Figure 3.5a where $p = a$. Otherwise, if $p = c$, then we can colour the 4-pole $M$ according to Figure 3.5b.




(a) when $p = a \neq c$ (b) when $p = c$

Figure 3.5: Colouring for case (iv)

□

If we use proper $(2,3)$-poles obtained from the Isaacs snarks in this construction, we create an infinite class of strictly critical snarks from $\mathcal{S}(5,6)$. Its smallest member (depicted in Figure 3.6) consist of 66 vertices and up to our knowledge, it is the smallest known strictly critical snark in $\mathcal{S}(5,6)$.

In this construction, the $(2,2,2)$-pole $\mathrm{TTT}_{sc}(T_1, T_2, T_3)$ can be combined with any odd $(2,2,2)$-pole. An interesting approach to how we can construct a larger odd $(2,2,2)$-pole with girth 6 from smaller ones shall be shown in Lemma 5.18. However, currently, we do not know under what conditions the resulted snark will be critical. It is very likely that we would have to put some restrictions on odd $(2,2,2)$-poles we could use as we could see in Definition 11.

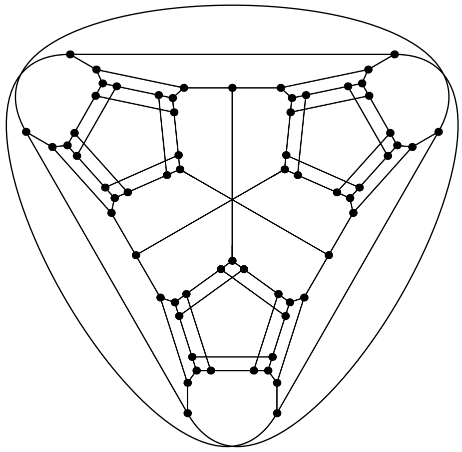


Figure 3.6: A strictly critical snark of order 66 from $\mathcal{S}(5, 6)$

## 3.3 Strictly critical snarks of cyclic connectivity 6

To construct cyclically 6-connected snarks, we shall use the *superposition* introduced by Kochol [13]. In superposition, we replace each vertex of a given graph $G$ by a *supervertex* which is a multipole with tree connectors and we replace each edge of $G$ by a *superedge*—a multipole with two connectors. After that, we perform junctions of corresponding connectors, hence they have to have the same width. Superposition is a far more general notion than we need, therefore we only describe one type of the superposition we shall use.

For a start, we define a specific type of a junction of two connectors we shall often use in this section. Consider that we have a $(i, c, r_1)$-pole $M(I, C_1, R_1)$ and $(c, o, r_2)$-pole $N(C_2, O, R_2)$. Inspired by the notation in [15], we define a *serial junction* $M \circ N$ of $M$ and $N$ as the $(i, o, r_1 + r_2)$-pole $P(I, O, R_1 \cup R_2)$ which arises by the junction of the connectors $C_1$, $C_2$ and by union of the connectors $R_1$ and $R_2$. In this case, we allow $r_1$ or $r_2$ to be zero and we treat a multipole $M(S_1, S_2, \emptyset)$ with an empty connector as the multipole $M(S_1, S_2)$.

For $k \geq 1$ and a multipole $M(S_1, S_2, \dots, S_k)$ where $|S_1| = |S_2|$, the *closure* $\overline{M}$ of $M$ is constructed by the junction of the connectors $S_1$, $S_2$.

We shall use the supervertex $V$ depicted in Figure 3.7 which contains two connectors of width 3 and one connector of width 1 in this order. For the superedge $E_5$, we remove the two non-adjacent vertices from the Isaacs snark $J_5$ as shown in Figure 3.8. For odd $k \geq 7$, the superedge $E_k$ is constructed by substituting the 6-pole $Y_2$ contained in $E_5$ by the 6-pole $Y_{k-3}$. We call these superedges *Isaacs superedges.*

Using the superedges and the supervertices defined above, we recursively define a *k-chain* as follows: A 1-chain is any $(3, 3, 1)$-pole of the form $F_0 \circ V \circ F_1$ where $F_0$ and $F_1$ are arbitrary Isaacs superedges. A $(k + 1)$-chain is a $(3, 3, k + 1)$-pole of the form $R_k \circ V \circ F_{k+1}$ for an arbitrary Isaacs superedge $F_{k+1}$ and an arbitrary $k$-chain

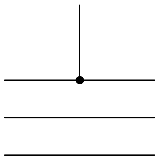


Figure 3.7: The supervertex $V$

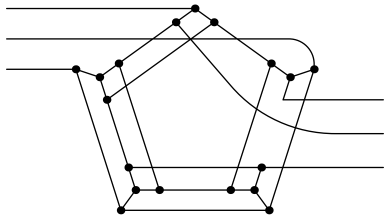


Figure 3.8: The superedge $E_5$

$R_k$. Then, we define a *supercycle of length* $k$, or for short a *supercycle*, as a $k$-pole of the form $SC_k = \overline{R_k \circ V}$ where $R_k$ is an arbitrary $k$-chain. The $k$-chains have colouring properties similar to properties of the $(k-1)$-path which is illustrated in the following lemma.

**Lemma 3.2.** *Let $R_k = F_0 \circ V_1 \circ F_1 \circ \cdots \circ V_n \circ F_k$ be an arbitrary $k$-chain and let $(i, o, r_1, \ldots, r_n)$ be a colouring of the dangling edges of the path of length $k-1$. Choose arbitrary $a$, $b$, $c$, $x$, $y$, $z$ such that $a+b+c=i$ and $x+y+z=o$. Then the colouring $(a, b, c, x, y, z, r_1, \ldots, r_n)$ is contained in the colouring set of $R_k$.*

*Proof.* We employ mathematical induction on $n$. For $n = 1$ and $F_0 = F_1 = E_5$, we verified using a computer that the $(3, 3, 1)$-pole $E_5 \circ V \circ E_5$ has the desired colouring set. Every other superedge $E_l$ can be obtained from $E_5$ by substituting the $(3, 3)$-pole $Y_2$ by the $(3, 3)$-pole $Y_{l-3}$. Since these $(3, 3)$-poles are colour equivalent, we have $\mathrm{Col}(F_0 \circ V \circ F_1) = \mathrm{Col}(E_5 \circ V \circ E_5)$.

Let us suppose that Lemma 3.2 holds for some $n = t$ and let $(i, o, r_1, \ldots, r_{t+1})$ be a colouring of the path of length $t+1$. Consider arbitrary $a$, $b$, $c$, $x$, $y$, $z \in \mathbb{K}$ such that $a+b+c = i$ and $x+y+z = o$. Suppose that $F_{t+1}((d_1, d_2, d_3), O_{t+1}) = J_{l_{t+1}} - (u_{t+1}, v_{t+1})$, denote the end vertex of the dangling edge $d_1$ by $w$. Since the proper $(2, 3)$-pole $T((d_1, d_4), (e_1, e_2, e_3)) = (J_{l_{t+1}} - u_{t+1}w) - v_{t+1}$ is perfect, it is colourable by a colouring $\varphi$ such that $\varphi(e_1, e_2, e_3, d_1, d_4) = (x, y, z, o+r_1, r_1)$. After removing the vertex $u_{t+1}$, we obtain the $(3, 3)$-pole $F_{t+1}$ coloured by the colouring $\varphi$ in such a way that $\varphi(d_1, d_2, d_3) = (o + r_1, p, q)$ where $p + q = r_1$. Then the semiedges in the first connector of $V_{t+1} \circ F_{t+1}$ are coloured by colours $(o, p, q)$. Finally, by the induction hypothesis, the $(3, 3, t)$-pole $R_t$ is colourable with its dangling edges coloured by $(a, b, c, o, p, q, r_1, \ldots, r_t)$ and after applying a junction, the $(3, 3, t + 1)$-pole $R_t \circ V_{t+1} \circ F_{t+1}$ is colourable by the desired colouring. □

**Lemma 3.3.** *For every $k \geq 2$ and every supercycle $SC_k$, $\mathrm{Col}(SC_k) = \mathrm{Col}(C_k)$.*

*Proof.* The inclusion $\mathrm{Col}(SC_k) \subseteq \mathrm{Col}(C_k)$ follows from the properties of a proper superposition [13]. Let $G(C_k(e_1, e_2, \ldots, e_k)) = v_1 v_2 \ldots v_k$ and $SC_k = \overline{R_k \circ V}$. Consider a colouring $\varphi(C_k) = (c_1, c_2, \ldots, c_k) \in \mathrm{Col}(C_k)$. The $(3, 3, 1)$-pole $V$ is colourable in such a way that flows through its connectors are $a = \varphi(v_{k-1}v_k)$, $b = \varphi(v_k v_1)$ and $c_k$. Since

$(b, a, c_1, c_2, \ldots, c_k)$ is a colouring of the $(k-1)$-path $v_1 v_2 \ldots v_{k-1}$, by Lemma 3.2, the $k$-chain $R_k$ admits desired colours on its semiedges and thus $\varphi(C_k) \in \mathrm{Col}(SC_k)$. □

**Theorem 3.2.** *Let $S = M * C_k$ be a critical snark with a $k$-cycle $C_k(e_1, e_2, \ldots, e_k) = v_1 v_2 \ldots v_k$ and let $SC_k$ be a supercycle of length $k$ consisting of superedges $F_j$ for $1 \leq j \leq k$. Then the snark $T = M * SC_k$ obtained by employing a superposition of the cycle $C_k$ in $S$ is critical.*

*Proof.* In this proof, we consider that

$$SC_k(r_1, r_2, \ldots, r_k) = \overline{F_1 \circ V_1 \circ F_2 \circ \cdots \circ V_{k-1} \circ F_k \circ V_k}$$

where for $1 \leq j \leq k$, $V_j$ is a copy of the supervertex $V$ and $F_j((i_1^j, i_2^j, i_3^j), (o_1^j, o_2^j, o_3^j)) = J_{l_j} - (u_j, t_j)$ for some odd $l_j \geq 5$. We shall use the denotation $i_\iota^j$, $o_\iota^j$ also for the corresponding links of the snark $J_{l_j}$. Furthermore, we denote the only vertex contained in $V_j$ by $w_j$. All these indices are taken modulo $k$. A scheme of the snark $T$ we shall use is depicted in Figure 3.9.

Let $x$, $y$ be arbitrary adjacent vertices in the snark $T$. To show that $T$ is a critical snark, we have to show that the 4-pole $T - [x, y]$ is colourable. The vertex $x$ can be taken from $k$-pole $M$, from some superedge contained in $SC_k$ or from some supervertex contained in $SC_k$. According to this, we divide the proof into several cases.

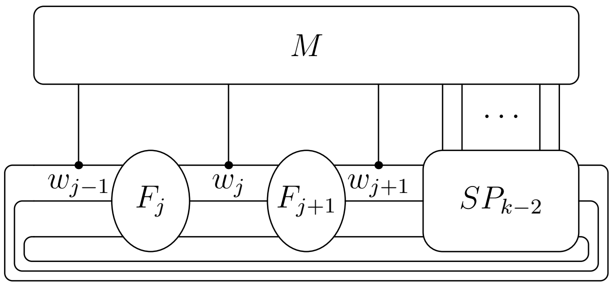



Figure 3.9

**Case (i)** Both the vertices $x$, $y$ belong to the $k$-pole $M$. Since the snark $S$ is critical the 4-pole $S - [x, y] = M * C_k - [x, y]$ is colourable and as the $k$-poles $C_k$ and $SC_k$ are colour equivalent, the multipole $M * SC_k - [x, y] = T - [x, y]$ is also colourable.

**Case (ii)** The vertex $x$ belongs to $M$ and $y = w_j$ for some $1 \leq j \leq k$. The $(2,2)$-pole $N((f_1, f_2), (f_3, f_4)) = S - [x, v_j]$ is colourable by some colouring $\varphi$ such that $\varphi(f_1) = \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = a$. For this colouring, we have $\varphi(v_{j-1} v_{j-2}) = p \neq a$, $\varphi(e_{j-1}) = p + a$, $\varphi(v_{j+1} v_{j+2}) = q \neq a$, $\varphi(e_{j+1}) = q + a$ for some (not necessary distinct) $p$, $q \in \mathbb{K}$. We colour the $(k+1)$-pole $SC_k - y$ as shown in Figure 3.10 where $\varphi(r_j) = \varphi(e_j)$ for $j \neq i$. From Lemmas 2.4 and 3.2, we know that the superedges $F_j$, $F_{j+1}$ and the chain $SP_{k-2}$ admit such colourings.

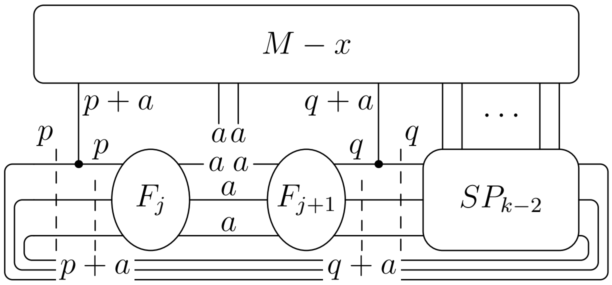



Figure 3.10: Colouring for the case (ii)

**Case (iii)** The vertex $x$ is identical with $w_j$ for some $1 \leq j \leq k$ and $y$ belongs to some superedge, say, $F_j$. The $(2,2)$-pole $N_1 = J_{l_j} - [y, t_j]$ is colourable by a colouring which assigns the same colour $q$ to all its dangling edges. When we remove the vertex $u_j$ from the $(2,2)$-pole $N_1$, we get the $(3,2,2)$-pole $E'_{l_j} = (J_{l_j} - u_j) - [y, t_j]$ with its dangling edges coloured by $(a, b, c, q, q, q, q)$ for $\{a, b, c\} = \mathbb{K}$. Since $S$ is a critical snark, the $(2,2)$-pole $N((f_1, f_2), (f_3, f_4)) = S - [v_{j-1}, v_j]$ is colourable by a colouring $\varphi$ such that $\varphi(f_1) = \varphi(f_2) = p \neq a$ and $\varphi(f_3) = \varphi(f_4) = q$ for not necessary distinct $p, q \in \mathbb{K}$. This colouring $\varphi$ assigns to the link $e_{j+1}$ a colour $r \neq q$ because the link $e_{j+1}$ and the semiedge $f_4 = (v_{j+1})$ are adjacent in $N$. Since the proper $(2,3)$-pole $(F_{j+1} - o_1^{j+1}) - u_{j+1}$ is perfect, it admits the colouring $(q+r, r, q, q, q)$ and hence the superedge $F_{j+1}$ admits a colouring $(q, q, q, q+r, c_1, c_2)$ for some $c_1, c_2 \in \mathbb{K}$ such that $c_1 + c_2 = r$. The colouring of the $(2,2)$-pole $M - [x, y]$ is depicted in Figure 3.11.

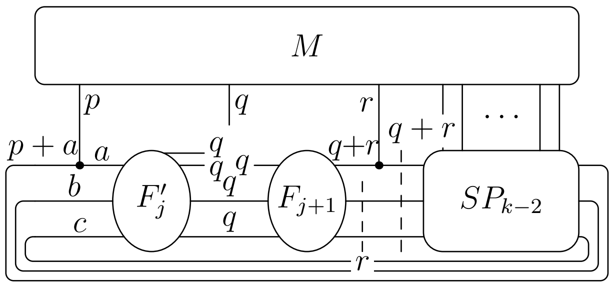



Figure 3.11: Colouring for the case (iii)

**Case (iv)** Both the vertices $x$, $y$ belong to the same superedge $F_j$ for some $j$. Then from a colouring of the $(2,2)$-pole $J_{l_j} - [x, y]$, we can obtain a colouring of the $(3,3,2,2)$-pole $(J_{l_j} - (u_j, t_j)) - [x, y]$ assigning to its first six dangling edges colours $(a, b, c, p, q, r)$ such that $a + b + c = p + q + r = 0$. Since the $(2,2)$-pole $S - [v_j, v_{j+1}]$ admits a colouring $(a', a', p', p')$ for some $a' \neq a$ and $p \neq p'$, we can colour the $(2,2)$-pole $T - [x, y]$ as shown in Figure 3.12. We obtained the colouring of $SP_{k-2} \circ V_{j-1} \circ F_{j-1} = SP_{k-1}$ from Lemma 3.2.

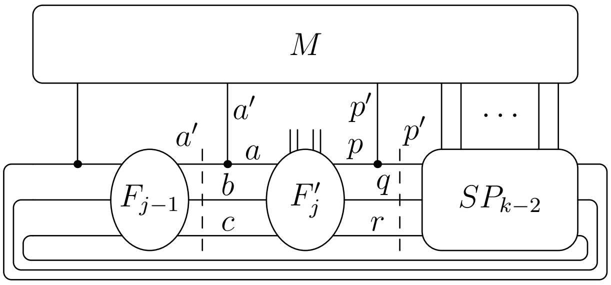



Figure 3.12: Colouring for the case (iv)

**Case (v)** The vertices $x$, $y$ belong to two different superedges, say $x \in V(F_j)$ and $y \in V(F_{j+1})$. The $(2,2)$-pole $N_1 = S - [v_j, v_{j+1}]$ is colourable by a colouring that assigns the same colour $a$ to all its semiedges. The link $e_{j-1}$ of $N_1$ has assigned a colour $b$ different from $a$ since it is incident with the dangling edge $(v_{j-1})$. Now, according to Lemma 2.4 the $(3,3)$-pole $N_2((i_1^j, i_2^j, i_3^j), (f_1, f_2, f_3)) = J_{l_j} - (u_j, x)$ is colourable by a colouring $\varphi$ such that $\varphi(i_1^j) = c = a + b$, $\varphi(i_2^j) + \varphi(i_3^j) = b$, $\varphi(f_1) = a = \varphi(o_1^j) + \varphi(o_2^j)$ and $\varphi(f_2) + \varphi(f_3) = 0$. If we remove the vertex $t_j$ along with its dangling edge $f_1$ from $N_2$, we get two more dangling edges $o_1^j$ and $o_2^j$ which have assigned the colours $b$, $c$ in some order in the colouring $\varphi$, so let $\varphi(o_1^j) = b'$ and $\varphi(o_2^j) = c'$. Furthermore, the $(2,2)$-pole $N_3((g_1, g_2), (g_3, g_4)) = J_{l_{j+1}} - [u_{j+1}, y]$ is colourable by a colouring $\psi$ such that $\psi(g_1) = \psi(g_2) = c'$ and $\psi(g_3) = \psi(g_4)$. The colouring $\psi$ assigns to the links $o_1^{j+1}$, $o_2^{j+1}$ and $o_3^{j+1}$ pairwise distinct colours $p$, $q$, $r$, respectively. We may assume that $p \neq a$ because if we had $\psi(o_1^{j+1}) = a$, then we could permute the colours $a$, $b' \neq a$ and get $\psi(o_1^{j+1}) \neq a$. Using all described colourings, the $(2,2)$-pole $T - [x, y]$ can be coloured as shown in Figure 3.13.

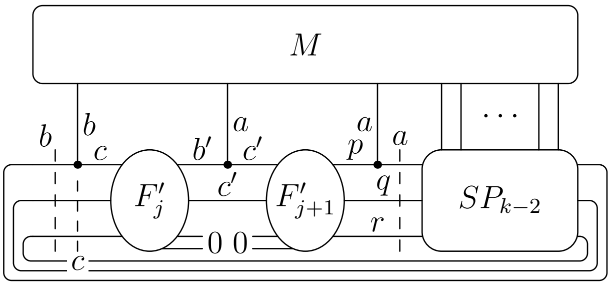



Figure 3.13: Colouring for the case (v)

□

For now, we are ready to describe an infinite class of strictly critical snarks with cyclic connectivity 6.

**Theorem 3.3.** *For each $n \geq 414$ such that $n \equiv 4 \pmod 8$, there exists a strictly critical cyclically 6-connected snark of order $n$.*

*Proof.* We take one of the 7 strictly critical snarks of order 36 consisting of three proper $(2,3)$-poles from the Petersen graph and one 6-cycle, and perform a superposition alongside the cycle of length 21 as shown in Figure 3.14. This eliminates all 5-cuts. By Theorem 3.2, the new snark is critical and as it still contains three pairs of removable vertices, it is not bicritical. When we use all superedges from $J_5$, we obtain a snark $S_{414}$ of order 414. A snark of order $414 + 8k$ is obtained when we replace the superedge $E_5$ by the superedge $E_{5+2k}$. □

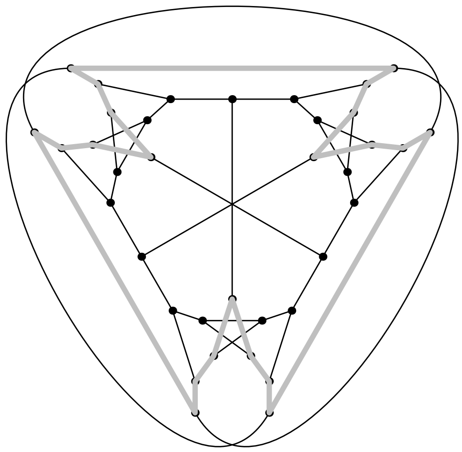


Figure 3.14: A cycle of length 21 to be replaced in a superposition

Up to our knowledge, the snark $S_{414}$ is the smallest strictly critical cyclically 6-connected snark. It is hard to believe that there is none such smaller snark, however, discovering it would probably require a new method of constructing cyclically 6-connected snarks. We leave it as a problem.

**Problem 3.1.** *Does there exist a strictly critical cyclically* 6*-connected snark of order smaller than* 414*?*

# Chapter 4

# An infinite class of bicritical snarks

In this chapter, we illustrate how we can use the structures observed in small bicritical snarks to construct infinite classes of bicritical snarks. We will focus on the infinite classes described in our bachelor thesis [19], namely on the most simple class. We aim to find conditions of used multipoles sufficient for the constructed snark to be bicritical.

As mentioned in [19, page 30], multipoles used in constructions of bicritical snarks need not be created from bicritical snarks. Obviously, it is necessary to eliminate all pairs of removable vertices from the used construction blocks, but generally, we do not understand under what conditions this is sufficient. Even if all the construction blocks were taken from bicritical snarks we cannot be sure that the resulting snark will be bicritical.

**Example 2.** Consider, for instance, the Goldberg-Loupekine snark $GL$ consisting of three Petersen negators arranged along a circle, with residual edges joined in an additional vertex (cf. Figure 4.1). We know that the snark $GL$ is bicritical and that it has three pairs of removable edges. Take one such pair $e$, $f = uv$ and replace the vertex $v$ and the edge $e$ with the colour-equivalent NT $(2, 3)$-pole $P_{\mathrm{NT}}$ consisting of a negator $N_P$ and a proper $(2, 3)$-pole $T_P$ constructed from the (bicritical) Petersen graph $P$ (see also Section 2.7). The resulting snark has order 38 and we have checked (with the help of a computer) that it is not bicritical despite the fact that all the construction blocks are taken from bicritical snarks.

The purpose of this chapter is to illustrate that imposing certain additional requirements on the multipoles can assure bicriticality of the resulting snark in a fairly general setting. The described requirements are not overly restrictive and it is even possible that most construction blocks taken from bicritical snarks (of any given order) satisfy them.

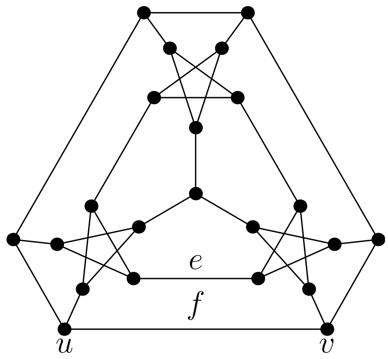



Figure 4.1: The Goldberg-Loupekine snark $GL$

## 4.1 Class NNN

For our demonstration, we have chosen snarks constructed by an NN-expansion (cf. [19, Section 3.2]). This class is perhaps the simplest of the described infinite classes, but a similar approach also works for the rest of them. For the purpose of proving irreducibility, we will view these snarks as consisting of three negators $N_i(I_i, O_i, r_i) = \mathrm{Neg}(S_i, u_i, v_i)$ for $i \in \{1, 2, 3\}$ arranged along a circle with an additional vertex attached to the residual semiedges (see Fig. 4.2). We denote the resulting graph by $\mathrm{NNN}(N_1, N_2, N_3)$.

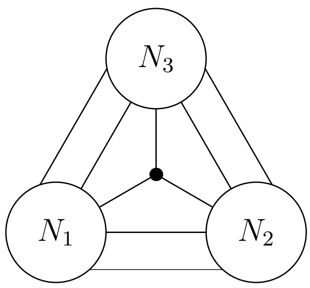



Figure 4.2: A schematic drawing of a snark $\mathrm{NNN}(N_1, N_2, N_3)$

As mentioned at the beginning of this chapter, restrictions are to be imposed on construction blocks, not on the snarks they originated from. We will call a negator $N = \mathrm{Neg}(S; u, v)$ *bicritical* if the 6-pole $S - (x, y)$ is colourable for every two distinct vertices $x$, $y \in V(N)$. The following theorem shows that this property is necessary.

**Theorem 4.1.** *Let $N_1$, $N_2$, $N_3$ be three negators such that $S = \mathrm{NNN}(N_1, N_2, N_3)$ is a bicritical snark. Then all the negators $N_1$, $N_2$, $N_3$ are bicritical.*

*Proof.* Consider one of the negators, say, $N_1 = N(S_1; u, v)$, and two of its vertices $x$, $y$. Since $S$ is a bicritical snark, the 6-pole $S - (x, y)$ is colourable. By replacing the 5-pole $\mathrm{NN}(N_2, N_3)$ with the colour-equivalent $P_2$, we get the 6-pole $S_1 - (x, y)$ which is thus also colourable. Hence the negator $N_1$ is bicritical. ☐

We do not know whether this property—or even the stronger assumption that all three negators are taken from bicritical snarks—is also sufficient. We have tested approximately 600,000 snarks of the class NNN constructed using at most two different negators. We used negators from bicritical snarks from $\mathcal{S}(5, 5, \leq 30)$. All of the tested snarks were bicritical.

In order to state a sufficient condition, we introduce the following rather technical property of negators (we will henceforth assume that the negator comes from a snark of girth at least 5 to avoid ambiguity of notation and certain corner cases).

**Definition 12.** A negator $N((i_1, i_2), (o_1, o_2), r) = \mathrm{Neg}(S; u, v)$ is called *good* if it satisfies both of the following properties:

(i) For any pair of vertices $x \in \{u, v\}$, $y \in V(N)$ and any pair of dangling edges $e$, $f$ of the 6-pole $S - (x, y)$ formerly incident with $x$, there exists a colouring $\varphi$ of the 6-pole $S - (x, y)$ such that $\varphi(e) = \varphi(f)$.

(ii) For any vertex $y \in V(N)$, there exist colourings $\varphi_1$ and $\varphi_2$ of the 8-pole $N - y$ satisfying

- $(\varphi_1(i_1), \varphi_1(i_2), \varphi_1(o_1), \varphi_1(o_1), \varphi_1(r)) = (a, a, b, b, a)$,
- $(\varphi_2(i_1), \varphi_2(i_2), \varphi_2(o_1), \varphi_2(o_1), \varphi_2(r)) = (a, a, b, b, b)$,

where $a, b \in \mathbb{K}$ and $a \neq b$.

Recall that we consider the multipole $M = S - (x, y)$ where $x \neq y$ always as a 6-pole even if the vertices $x$, $y$ are adjacent—in such a case the edge $xy$ from $S$ and remains in $M$ as an isolated edge. Observe that in this case, the property (i) is always satisfied. The 6-pole $M$ is colourable by the irreducibility of $S$ and by the Parity Lemma, we get that the two dangling edges formerly incident with $x$ have the same colour—this colour can be assigned to the one isolated edge in $M$, so all three semiedges formerly incident with $x$ have the same colour for some colouring of $M$

If we consider an bicritical snark $S$, the 6-pole $M = S - (x, y)$ is colourable for any removed different vertices $x$, $y \in V(S)$. By Parity Lemma, we know that in every colouring of $M$, two semiedges formerly incident with $x$ have to have the same colour. The property (i) enables us to choose these two semiedges arbitrary. We tested all bicritical snarks from $\mathcal{S}(5, 5, \leq 36)$ and only six of them can be used to create a negator violating the condition (i). We describe them in Section 4.2. Also, we tested the same snarks for the property (ii). Although, there are much more negators violating the condition (ii), on average, about 90 percent of removed paths of length two from some bicritical snark from $\mathcal{S}(5, 5, \leq 36)$ yields a good negator.

Now, we are ready to state sufficient condition.

**Theorem 4.2.** *Let $N_1, N_2, N_3$ be three good bicritical perfect negators. Then $S =$ $\mathrm{NNN}(N_1, N_2, N_3)$ is a bicritical snark.*

*Proof.* For $i \in \{1, 2, 3\}$, assume that the negator $N_i$ was constructed from a snark $S_i$ by removing a path $u_i w_i v_i$. We will denote the two connectors of the negator $N_i$ by $I_i = (i_{i1}, i_{i2})$ and $O_i = (o_{i1}, o_{i2})$ and its residual edge by $r_i$. Let $x$, $y$ be two arbitrary vertices of the snark $S$. We shall show that the 6-pole $S - (x, y)$ is colourable and hence $S$ is bicritical.

**Case (i)** If both vertices $x$, $y$ belong to the same negator $N_i$, we can replace the other two negators with the colour-equivalent $P_2$ (path of length two), completing the negator $N_i$ to the snark $S_i$. Since $N_i$ is bicritical, $S_i - (x, y)$ is colourable, hence so is $S - (x, y)$.

**Case (ii)** Let the vertices $x$, $y$ belong to two different negators, let us say that $x \in V(N_1)$ and $y \in V(N_2)$. Remove the vertices $v_1$ and $x$ from the snark $S_1$ and denote the semiedges formerly incident with $v$ by $e_1$, $e_2$, $e_3$ so that $e_3$ is incident with $w_1$. According to the property (i) of the good negator $N_1$, there exists a colouring $\varphi_1$ of the 6-pole $S_1 - (v_1, x)$ such that $\varphi_1(e_1) = \varphi_1(e_2) = p$. Let $\varphi_3(e_3) = a$ and $\varphi_1(u_1 w_1) = b \neq a$. We can simply restrict the colouring $\varphi_1$ to a colouring of the multipole $N_1 - x$ in which $\varphi_{1\Sigma}(I_1) = \varphi_1(u_1 w_1) = b$, $\varphi_1(r_1) = a + b$ and $\varphi_{1\Sigma}(O_1) = 0$.

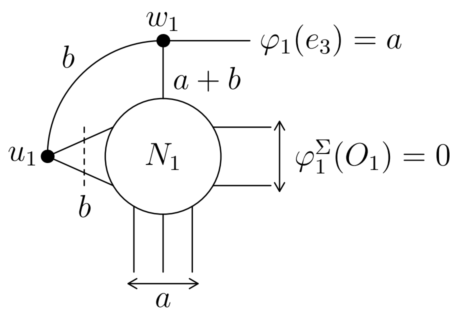



Figure 4.3: The colouring $\varphi_1$ of $N_1$

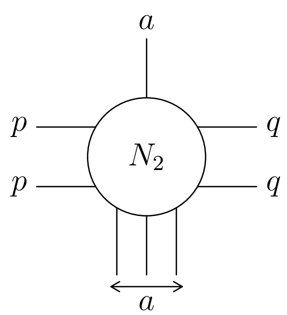



Figure 4.4: The colouring $\varphi_2$ of $N_2$

By the property (ii) of the good negator $N_2$, there exists a colouring $\varphi_2$ of $N_2 - y$ such that $\varphi_2(i_{21}) = \varphi_2(i_{22}) = p$, $\varphi_2(r_2) = a$ and $\varphi(o_{21}) = \varphi(o_{22}) = q \neq p$. Note that this can be always achieved. When $p = a$, we choose $q = b$ and when $p \neq a$, we choose $q = a$.

Now, we can unite the multipoles $N_1 - x$, $N_2 - y$ and perform junctions of the connectors $O_1$ and $I_2$ and joining the semiedges $r_2$ and $r_1$ with a new vertex. We obtain the multipole $M = \mathrm{NN}(N_1, N_2) - (x, y)$ coloured by a colouring $\varphi$ which can be easily obtained from the colourings $\varphi_1$ and $\varphi_2$ (see Fig. 4.5).

Finally, we unite the negator $N_3(I_3, O_3; r_3)$ with $M$ and perform junctions of connectors $I_3$ and $O_2$, $O_3$ and $I_1$, $r_3$ and the residual edge $r$ of the multipole $M$.

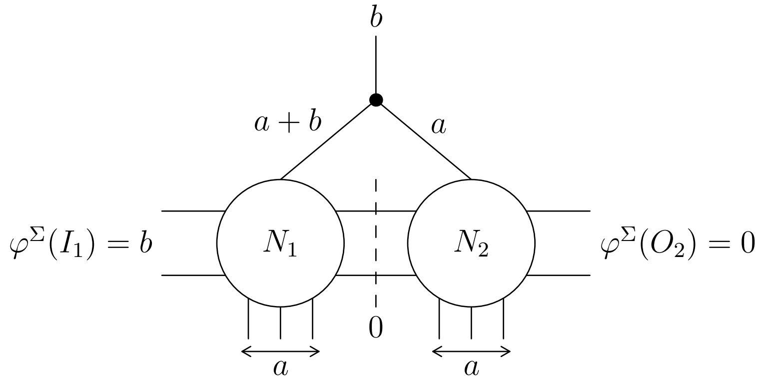



Figure 4.5: Colouring of the 5-pole

Observe that $\varphi_{\Sigma}(O_2) = 0$, $\varphi_{\Sigma}(I_1) = \varphi(r) = b \neq 0$. Since the negator $N_3$ is perfect, it admits flows 0, $b$ and $b$ through its connectors, so the obtained 6-pole $S - (x, y)$ is colourable.

**Case (iii)** Let the vertex $y$ belongs to none of the negators, so it is the vertex connecting the residual semiedges from all three negators $N_1$, $N_2$, $N_3$. Let $x \in V(N_1)$. We take the colouring $\varphi_1$ of $N_1 - x$ such that $\varphi_1(i_{11}) = \varphi_1(i_{12}) = a$ and $\varphi_1(r_1) = \varphi(o_{11}) = \varphi_1(o_{12}) = b \neq a$. Since $N_2$ is a perfect negator, it admits a colouring $\varphi_2$ such that $\varphi_{2\Sigma}(I_2) = 0$, $\varphi_2(r_2) = \varphi_{2\Sigma}(O_2) = b$. Further, we proceed similarly as in Case 2: we join all three negators and get a colouring of the 6-pole $S - (x, y)$. □

In order to create an infinite class of bicritical snarks, we need an infinite family of good negators. As one could expect, negators constructed from Isaacs snarks are good. Recall that each pair of non-adjacent edges of the Isaacs snark is essential (cf. Section 2.6).

**Lemma 4.1.** *For every odd $k \geq 5$, every negator $N$ constructed from the snark $J_k$ is good.*

*Proof.* Let $x$, $y$ be arbitrary non-adjacent vertices of the Isaacs snark $J_k$ for an odd $k \geq 5$ and let $e_1$, $e_2$, $e_3$ be the edges incident with $x$. There always exists an edge $f$ incident with $y$ that is not adjacent to $e_3$. The pair of edges $\{e_3, f\}$ is essential in $J_k$ which means that the multipole $(S - \{e, f\})$ with $x$ suppressed has a colouring $\varphi$. If we cut the edge arisen from the suppression of $x$ into two dangling edges corresponding to $e_1$ and $e_2$ and remove $y$, we get the $(3, 3)$-pole $J_k - (x, y)$ with a colouring in which the dangling edges corresponding to $e_1$, $e_2$ have the same colour. By a suitable choice of $e_3$, we can obtain the same colour on any two dangling edges formerly incident with $x$. This holds for an arbitrary choice of non-adjacent $x$, $y \in J_k$ and the same holds trivially for any adjacent $x$, $y$, thus every negator constructed form $J_k$ satisfies the property (i).

The fact that these negators also satisfy the property (ii) will be proved by induction on $k$. For the basis, we checked by a computer that all the negators derived from the snarks $J_5$, $J_7$ and $J_9$ satisfy the property (ii).

Consider the Isaacs snark $J_k$ for an odd $k \geq 11$. Remove an arbitrary path $uwv$ from $J_k$ and an arbitrary vertex $x$ from the negator $N = \mathrm{Neg}(J_k; u, v)$; denote the resulting 8-pole $M$ and the dangling edges corresponding to the dangling edges of the negator $N$ by $i_1$, $i_2$, $o_1$, $o_2$ and $r$ in the usual way. The path $uwv$ intersects at most three consecutive copies of the Isaacs $(3,3)$-pole $Y$ and the removal of the vertex $x$ corrupts at most one other copy of $Y$. Consequently, there are at least four intact consecutive copies of $Y$ in $M$; let us denote $Y_4$ the $(3,3)$-pole they induce. We replace them with a $(3,3)$-pole $Y_2$ consisting of two copies of $Y$ and denote the resulting multipole $M'$. Clearly, $M'$ is isomorphic to the multipole obtained from $J_{k-2}$ by removal of a certain path of length two and a certain additional vertex. By the induction hypothesis, there exists a colouring of $M'$ in which the dangling edges corresponding to $i_1$, $i_2$, $o_1$, $o_2$, $r$ have colours exactly as desired for either $\varphi_1$ or $\varphi_2$ from the property (ii). Since the multipoles $Y_4$ and $Y_2$ are colour-equivalent (Lemma 2.2), the desired colours can also be assigned to the semiedges $i_1$, $i_2$, $o_1$, $o_2$, $r$ of $M$. Hence, any negator constructed from the Isaacs snark $J_k$ satisfies the property (ii). □

**Theorem 4.3.** *Let $k$, $l$, $m$ be three odd integers greater or equal* 5*. Moreover, let $N_1$, $N_2$, $N_3$ be negators from the snarks $J_k$, $J_l$, $J_m$, respectively such than none of those negators contains* 5*-cycle. Then $S = \mathrm{NNN}(N_1, N_2, N_3)$ is a bicritical snark with girth* 6 *and cyclic connectivity* 5.

*Proof.* The bicriticality of the snark $S$ follows from Lemma 4.1 and Theorem 4.2. Suppose that the snark $S$ contains a cycle $C_g$ of length $g < 6$. Since all the negators $N_1$, $N_2$, $N_3$ have girth 6, the vertices of $C_g$ belong to at least two negators of the snark $S$. Because $g < 6$, one negator $N$ contains exactly two of the vertices $u$, $v$ of $C_g$. The vertices $u$, $v$ are adjacent and they are incident with one dangling edge of $N$ each. When we complete the negator $N$ to a snark $J_n$, $n \in \{k, l, m\}$, the edge $uv$ would be contained in a triangle which a contradiction, since the snark $J_n$ has girth at least 5.

Since Isaacs snarks $J_k$ for odd $k \geq 5$ do not contain cuts of size smaller than 5, the snark $S$ has cyclic connectivity 5. □

When we use negators $N_{17}$ from the smallest admissible Isaacs snark $J_5$ (we choose the removed path of length two in such a way that it contains at least one vertex from the only 5-cycle in $J_5$), we get a snark $S_{52} = \mathrm{NNN}(N_{17}, N_{17}, N_{17}) \in \mathcal{S}(5, 6)$ which has 52 vertices (it is depicted in Figure 4.6). Up to our knowledge, it is the smallest known snark in $\mathcal{S}(5, 6)$ except the double star snark.

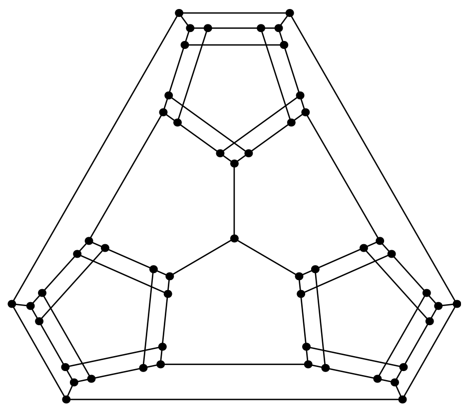


Figure 4.6: The second smallest known snark from $\mathcal{S}(5, 6)$

## 4.2 Non-removable edges which are not essential

As promised in the previous section, we are going to take a detailed look at snarks containing negators violating the condition (i) in Definition 12. All six known such snarks belong to the class NNN; they consist of two Petersen negators and one negator from a reducible snark of order 24 depicted in Figure 4.7 where the removed path of length 2 is dashed.

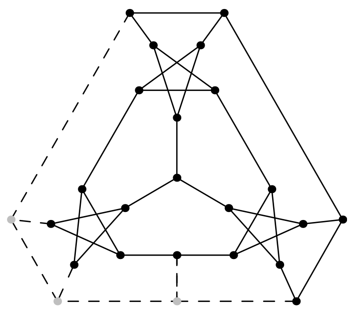


Figure 4.7

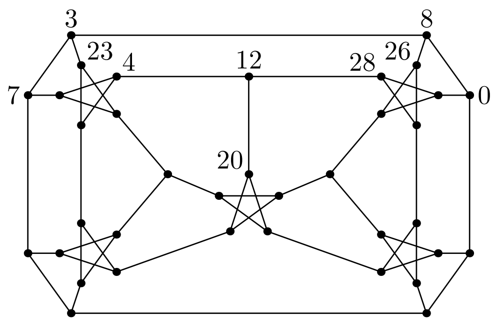



Figure 4.8: The snark $S_{36}$

One of these snarks, denoted by $S_{36}$, is illustrated in Figure 4.8. If we remove the pair of vertices 12, 3 (or 12, 8), we get a 6-pole $M$ such that for each colouring of $M$, the dangling edges incident with the vertices 20, 4 (or 20, 28) have different colours (this property has been verified by exhaustive computer search). When we construct a negator from the snark $S_{36}$ by removing a path of length two starting from the vertex 12, it violates even the weak version of the good negator condition in Definition 12.

The snark $S_{36}$ has another interesting property. If we take the 6-pole $S_{36} - \{3, 12\}$ and perform a junction of the semiedges (4) and (20), we get a 4-pole that is uncolourable (because the two joined semiedges have different colours in any possible 3-edge-colouring). Furthermore, we can add one vertex incident with semiedges (8) and (23); the resulting uncolourable multipole is isomorphic with $(S_{36} - \{\{12, 28\}, \{3, 7\}\}) \sim 12$. This implies that the pair of edges $\{12, 28\}, \{3, 7\}$ is not essential in $S_{36}$. This pair of edges is non-removable, as we verified with the help of a computer. This solves Problem 5.7 proposed by Chladný and Škoviera in [6] by showing that there exists a pair of non-removable edges in an bicritical snark which is not essential. The same holds for the pairs of edges $\{\{12, 28\}, \{3, 23\}\}$, $\{\{12, 4\}, \{8, 0\}\}$ and $\{\{12, 4\}, \{8, 26\}\}$.

# Chapter 5

# Small snarks with girth 6

Brinkmann et al. generated all snarks from $\mathcal{S}(4, 6, \leq 38)$ [4]. Aim of this chapter is to generate all non-trivial snarks from $\mathcal{S}(\leq 5, 6, 40)$. At present, there are not so many known small cyclically 6-connected snarks—only the Isaacs snarks $J_k$ for odd $k \geq 7$. Except them, the smallest known snark from $\mathcal{S}(6, 6)$ has order 118 and was constructed by Kochol in [12]. Therefore, finding another small snark from $\mathcal{S}(6, 6)$ would require a different approach and thus we do not hope to find all cyclically 6-connected snarks of order 40. However, snarks with smaller cyclic connectivity can be found thanks to the decompositions theorems by Goldberg [10] and Cameron et al. [18]. Note, there is no snark in $\mathcal{S}(1, 7, \leq 42)$ [3].

Some snarks from $\mathcal{S}(4, 6, 40)$ have already been generated and are available at [3]; all of them have cyclic connectivity 4. Recall that in Section 4.1 we stated that the smallest, up to our knowledge, known snark in $\mathcal{S}(5, 6)$ has order 52 except the double star snark of order 30. Small snarks with girth 6, mainly those with a little number of small cuts, can provide us a good base for constructions of cyclically 6-connected snarks. At the end of this chapter, we present several ideas which can be used to construct cyclically 6-connected snarks.

Since we will work with snarks with small cyclic connectivity, it is useful to recall that for $k \leq 2$ and any cubic graph $G$ one has that $\lambda(G) = k$ if and only if $\lambda_c(G) = k$, where $\lambda(G)$ is the edge-connectivity of a graph $G$. Also, a 3-edge-connected cubic graph is also cyclically 3-connected. Therefore, we shall only say $k$-connected when we shall deal with connectivity $k \leq 3$.

## 5.1 Generation of snarks with cyclic connectivity 3

As it will appear later, we shall use snarks with cyclic connectivity 3 in our constructions. A list of these snarks up to 24 vertices can be obtained from the list of all cubic graphs with girth 3 up to order 24 available at [3] by filtering uncolourable cyclically

3-connected ones. We know that every snark containing a 3-cut (and no smaller cut) is constructed by removing a vertex from a snark in $\mathcal{S}(3,3)$ and performing a junction with some 3-pole $M$. The 3-pole $M$ can be completed to a cubic graph by adding one vertex, hence by removing a vertex from an arbitrary cubic graph, we can reconstruct it back—we only need to check if it contains no small cut.

To generate all snarks from $\mathcal{S}(=3,3,n)$, we tried to combine all pairs of a snark $S \in \mathcal{S}(3,3)$ of order at most $n-2$ and a cubic graph $G$ of order at most $n-10+1+1 = n-8$ (because the smallest such snark is Petersen graph of order 10). For each $u \in V(S)$ and $v \in V(G)$ we generated six snarks of the form $(S-u)*(G-v)$, one for each ordering of semiedges in the 3-pole $S-u$. Then we filtered cyclically 3-connected and non-isomorphic snarks.

We were able to generate all 253,088,654 cyclically 3-connected snarks up to order 30. We expect that the number of snarks in $\mathcal{S}(=3,3,32)$ is over 2 milliards (about 150 GB) and that the class $\mathcal{S}(=3,3,32)$ contains at least 20 milliards of snarks. That would be hard to generate and also difficult to process in further constructions. Reaching order 34 is currently impractical since we would need cubic graphs of order 26 which are not readily available.

## 5.2 Cyclic connectivity 5

Suppose that we have a snark $S \in \mathcal{S}(=5,6)$, in other words, we can represent the snark $S$ as a junction of two 5-poles $M_1 * M_2$. Then according to [18] either one of the multipoles $M_1$, $M_2$ is uncolourable, or both of them can be completed to a snark by performing a junction with one of the following multipoles: the 5-cycle $C_5$, the path of length two $P_2$ or the multipole $M_{ev}$ consisting of one isolated edge and one vertex incident with three dangling edges. Denote the snarks the multipoles $M_1$, $M_2$ can be completed to by $S_1 = M_1 * N_1$, $S_2 = M_2 * N_2$.

In the beginning, we state a few useful lemmas.

**Lemma 5.1.** *If $N_i = M_{ev}$ then the snark $S_i$ is 3-connected or it is 2-connected with only one 2-cut that separates a parallel edge (a 2-cycle).*

*Proof.* Suppose that $\lambda_c(S_i) \leq 2$. Let $S_i$ be a junction of two $k$-poles $C_1$, $C_2$ with minimal possible $k$. Denote the removed edge by $e$ and the removed vertex by $v$. Without loss of generality, let $v \in V(C_1)$. The edge $e$ has to be a link of $C_2$. Otherwise, the multipole $C_2$ would be separable in $S$ by at most 2 edges.

If $k = 1$, then both of the multipoles $C_1 - v$ and $C_2 - e$ have at most 4 semiedges. Because $M_i = (S_i - e) - v$ contains a cycle, at least one of the multipoles $C_1 - v$ and $C_2 - e$ contains a cycle which implies $\lambda_c(S) \leq 4$. However, when $k = 2$, then the multipole $C_2 - e$ contained in the snark $S$ has 4 dangling edges. Therefore, the 4-pole

$C_2 - e$ has to be acyclic. This leaves us the only possibility—it consists of two adjacent vertices. Thus the component $C_2$ is a parallel edge.

Suppose that the snark $S_i$ contains another pair of parallel edges $f$, $g$. Since the snark $S$ contains no parallel edges, the edges $f$, $g$ has to be incident with $v$. However, then the snark $S$ would contain a 4-cut as shown in Figure 5.1.

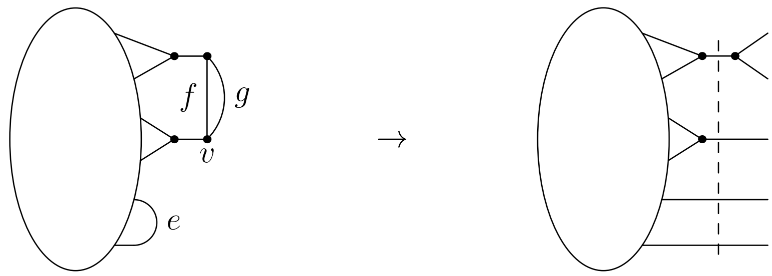



Figure 5.1

□

**Lemma 5.2.** *If $N_i = P_2$ then the snark $S_i$ is 3-connected.*

*Proof.* Suppose that the snark $S_i$ has a cycle separating $k$-cut $S_i = C_1 * C_2$ for some $k \leq 2$. Consider the multipoles $C_1 - P_1$ and $C_2 - P_2$ obtained by removing $P_2$ from the snark $S_1$ and denote the number of links between multipoles $C_1 - P_2$ and $P_2$, $C_2 - P_2$ and $P_2$, $C_1 - P_2$ and $C_2 - P_2$ by $x$, $y$, $z$, respectively. Additionally, we can assume that $x \geq y$ due to symmetry. If $y = 0$, then the multipole $C_2$ containing a cycle would be separable in the snark $S$ by $x \leq 2$ edges—a contradiction. Now, we know that the $k$-cut $S_1 = C_1 * C_2$ has to contain at least one link of $P_2$, hence $z \leq 1$. If $y = 1$, then the multipole $C_2 - P_2$ is separable by $z + y \leq 2$ edges in $S$, so $\lambda_c(S) = \lambda(S) \leq 2$ which is again a contradiction.

Therefore, $y = 2$ and $x = 3$. However, the multipoles $C_1 - P_2$ and $C_2 - P_2$ are connected with at most one edge in the snark $S$, so one of them has to contain a cycle and that multipole is separable with at most $x + z \leq 4$ edges—a contradiction. Thus, the snark $S_i$ is cyclically 3-connected. □

**Lemma 5.3.** *If $N_i = C_5$ then the snark $S_i$ is cyclically 4-connected.*

*Proof.* The proof remains similar to the proof of Lemma 5.2. The only difference is that since the 5-cycle is 2-edge-connected, then when it is contained in the cut $S_i = C_1 * C_2$, at least two of its edges belong to the cut. So, $k \leq 3$ would imply $z \leq 1$.

□

**Lemma 5.4.** *Let $M$, $N$ be two $k$-poles each containing a cycle such that the graph $M * N$ has girth 6. Then the minimal order $m(k)$ of the multipole $M$ (and also $N$) is for small values of $k$:*

| $k$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $m(k)$ | 14 | 13 | 12 | 11 | 6 |

*Proof.* We have done the proof by exhaustive computer verification of all cubic $k$-poles for desired values of $k$. □

**Lemma 5.5.** *There is no uncolourable* 5*-pole with girth* 6 *and order* 29 *or less.*

*Proof.* Suppose that there is such 5-pole $M$. Then $T = M * C_5$ is a snark of order at most 34. According to Lemma 5.3, we have $\lambda_c(T) \geq 4$. We tried to remove every 5-cycle from every snark of $\mathcal{S}(4, 4, \leq 34)$ using a computer. We determined the girth of each 5-pole arisen after a removal of $C_5$ and if it was at least 6, we checked whether it was colourable. This test found no desired 5-poles. □

**Lemma 5.6.** *Let both of the* 5*-poles $M_1$, $M_2$ are colourable, $N_1 = C_5$ and $N_2 \in \{P_2, M_{ev}\}$. Then the graph $M_1 * M_2$ is colourable.*

*Proof.* We use the graphs $X_1$, $X_2$ defined in the proof of Theorem 2 in [18]. The graph $X_1$ corresponding to the colouring set of $M_1$ consists of one 5-cycle. Since the graph $X_2$ contains a cycle and it has not length 5, the graphs $X_1$, $X_2$ share at least one edge and thus, the graph $M_1 * M_2$ is colourable. □

**Lemma 5.7.** *If both* 5*-poles $M_1$, $M_2$ are colourable and the order of $M_1 * M_2$ is at most $n$, then each of them was obtained by one of the following ways:*

*(i) Remove a vertex $v$ and split an edge not incident with $v$ in a snark from $\mathcal{S}(3, 3, \leq n - 16)$.*

*(ii) Choose a snark $S \in \mathcal{S}(3, 3, \leq n-14)$, an edge $e \in E(S)$ and a vertex $v \in V(S)$ not incident with $e$. Subdivide the edge $e$ with two additional vertices each incident with one dangling edge and remove the vertex $v$ from the snark $S$.*

*(iii) Remove a path of length two in a snark from $\mathcal{S}(3, 3, \leq n - 14)$.*

*(iv) Remove a* 5*-cycle in a snark from $\mathcal{S}(4, 4, n - 10)$.*

*Proof.* Ignoring the requirements of order, the lemma is the consequence of the Cameron's decomposition theorem [18] and Lemmas 5.1, 5.2 and 5.3. Using a computer search, we found that the smallest 5-poles of girth 6 constructed in the ways (i), (ii), (iii) and (iv) have orders 19, 21, 17 and 15, respectively. If $M_1 = S_1 - C_5$, then the 5-pole $M_2$ has at least 15 vertices, so $|M_1| \leq n - 15$ and $|S_1| \leq n - 10$. Moreover, by Lemma 5.6, if the 5-pole $M_1$ is constructed by (i), (ii) or (iii), then the 5-pole could not be constructed by (iv) and hence $|M_2| \geq 17$ and $|M_1| \leq n - 17$ which gives us desired upper bounds for order of the snark $S_1$. □

Now, we are ready to describe an algorithm to find all snarks of order 40. According to Lemma 5.4, the minimal size of the 5-poles $M_1$, $M_2$ is $40 - 11 = 29$ and all such 5-poles are colourable by Lemma 5.5. Hence, it is sufficient to consider only the case when both of the multipoles $M_1$ and $M_2$ are colourable.

Initially, we construct all 5-poles as described in (i), (ii), (iii) and (iv) of Lemma 5.7 and retain only those with girth 6 and no cycle separating cut of size 4 or smaller. Denote the set of the multipoles constructed according to (i), (ii), (iii) by $A$ and the set of the multipoles constructed according to (iv) by $B$. For each two multipoles $M_1$, $M_2$ from the set $A$ and each of the $5! = 120$ orderings of the dangling edges of $M_1$, we generate a graph $G = M_1 * M_2$ and if $G$ is uncolourable, $g(G) = 6$ and $\lambda_c(G) = 5$, then we keep it. In the end, we filter non-isomorphic snarks from the set of generated graphs. We do the same for each pair of multipoles $M_1$, $M_2$ from the set $B$. Note that it is required to test the colourability of generated snarks.

Results of the described algorithm show that the class $\mathcal{S}(=5, 6, 40)$ contains no snarks and we have not found any such snarks on 42 or 44 vertices from incomplete sets of larger multipoles. Remark that the set $B$ consists of only one 5-pole $J_5 - C_5$. Hence there is only one snark that can be constructed using the multipoles from the set $B$ and it is the Double Star snark.

## 5.3 Cyclic connectivity 4

Foremost, we define a fundamental operation used in constructions of snarks with cyclical connectivity 4 introduced by Isaacs [11]. Let $S_1$, $S_2$ be two snarks. A graph of the form $(S_1 - (e, f)) * (S_2 - [u, v])$, for some non-adjacent edges $e, f \in E(S_1)$ and adjacent vertices $u, v \in V(S_2)$ is called a *dot-product* of $S_1$, $S_2$ and it is always a snark.

Consider a snark $S \in \mathcal{S}(=4, 6)$. Since it contains a 4-cut, it can be represented as a junction $S = M_1 * M_2$ of two 4-poles $M_1$, $M_2$. According to Goldberg's decomposition theorem [10], either one of the 4-poles is uncolourable or the snark $S$ is a dot-product of two smaller snarks $S_1$, $S_2$. One of the 4-poles is completed by the 4-pole $L$ consisting of two vertices linked by a link, each of them is incident with two semiedges. The other 4-pole is completed by the 4-pole $R$ consisting of two isolated edges.

**Lemma 5.8.** *There is no uncolourable* 4*-pole with girth* 6 *and order at most* 28*.*

*Proof.* Suppose that there is a cyclically 4-connected snark $S = M_1 * M_2$ such that the 4-pole $M_2$ is uncolourable. According to Lemma 7 published by Andersen et al. in [1], the 4-pole $M_2$ can be completed to a cyclically 4-connected cubic graph $S_2 = L * M_2$ by adding two adjacent vertices. Since $M_2$ is uncolourable, $S_2$ is a snark. Henceforth, every uncolourable 4-pole $M_2$ can be obtained by removing two adjacent vertices from some snark from $\mathcal{S}(4, 4, \leq 30)$. After an exhaustive computer search, we found no uncolourable 4-poles of order up to 28.

□

**Lemma 5.9.** *If $S_i = M_i * R$, then the snark $S_i$ is 2-connected.*

*Proof.* To the contrary, suppose that $S_i$ has a bridge dividing the snark $S_i$ into components $C_1$ and $C_2$. Each of these components has to contain one edge of the 4-pole $R$. Therefore a cycle of the 4-pole $M_i$ is separable by at most 3 edges which contradicts the fact that $\lambda_c(S) = 4$. □

**Lemma 5.10.** *If $S_i = L * M_i$ then the snark $S_i$ is 3-connected.*

*Proof.* Let $e$ be a dangling edge of $M_1$ and $v$ its only end vertex. Then we can remove the vertex $v$ along with its dangling edge $e$ from the $(2, 2)$-pole $M_i$ and we get a $(2, 2, 1)$-pole isomorphic to $S_i - P_2$ which still has girth at least 6. Therefore, Lemma 5.2 implies that $\lambda_c(S_i) \geq 3$. □

**Lemma 5.11.** *If both of the 4-poles $M_1$, $M_2$ are colourable and the order of $M_1 * M_2$ is at most $n$, then one of them was obtained by the method (i) and the other by the method (ii):*

*(i) Remove two vertices along with link between them from a snark from $\mathcal{S}(3, 3, \leq n - 18)$.*

*(ii) Split two arbitrary links of some snark in $\mathcal{S}(2, 2, \leq n - 20)$.*

*Proof.* The smallest 4-poles with girth 6 which arise by the constructions (i) and (ii) have order 18 and 20, respectively. We verified this using a computer. The fact that both of the 4-poles $M_1$, $M_2$ cannot be constructed by the same method follows from Goldberg's decomposition theorem [10]. If $M_1$ is constructed by the method (i) and $M_2$ by method (ii), then since $M_1$ has at lest 18 vertices, $M_2$ has at most $n - 18$ vertices. Similarly, we obtain that $M_1$ has at most $n - 20$ vertices. □

Finally, we present an algorithm to find all snarks of the class $\mathcal{S}(= 4, 6, 40)$. For each snark $S \in \mathcal{S}(2, 2, \leq 22)$, we try to split every pair of non-adjacent edges $e, f \in E(S)$. If the 4-pole $S - (e, f)$ has girth 6 and no cut of size less than 4, then we add the 4-pole $S - (e, f)$ into a set $A$. Similarly, for each snark $T \in \mathcal{S}(3, 3, 20)$ and every its pair of adjacent vertices $u$, $v$ we determine the girth of the 4-pole $T - [x, y]$. If it is at least 6 and the 4-pole contains no cut of size smaller than 4, we add the 4-pole $T - [x, y]$ into a set $B$. Afterwards, for each 4-poles $M_1 \in A$ and $M_2 \in B$ and every ordering of the semiedges of $M_1$, we add the graph $M_1 * M_2$ to a set of constructed snarks $C$. Finally, we filter from the set $C$ only snarks that are mutually non-isomorphic.

By Lemma 5.8, it is sufficient to consider only the latter case of the decomposition theorem. Then, the correctness of the algorithm follows from Lemma 5.11. Note that we do not need to check the uncolourability of the generated snarks since they are dot-product of snarks which are guaranteed uncolourable.

By implementing this algorithm, we found out that the class $\mathcal{S}(=4, 6, 40)$ consists of 276 snarks.

## 5.4 Conclusion and greater orders

Summarising the results of the previous two section, we can say the following.

**Theorem 5.1.** *There are* 276 *non-trivial snarks of order* 40*, girth* 6 *and cyclic connectivity at most* 5*.*

We believe that our ideas can be used to generate snarks with girth 6 and order 42 or even 44. Currently, the greatest drawback is Lemma 5.5 since there are about 400 millions of cyclically 4-connected snarks of order 36 and it is currently computationally infeasible to generate the complete list of such snark for order 38. A solution to this problem is to complete the considered uncolourable 5-poles to a snark in a different way, most likely by a path of length 2. Yet we also need to improve the requirements for the snarks in Lemma 5.2. Although the bound 3 for the cyclic connectivity is best possible, small cuts are allowed only around the removed path of length 2. Therefore, it is possible to require that the snark $S_i$ is cyclically 4-connected after excluding several special cases. On the other side, it requires more precise case analysis and a care for technical details. We aim at generating snarks from $\mathcal{S}(4, 6)$ of higher orders in our further research.

## 5.5 Construction methods

Negators and proper $(2, 3)$-poles have very useful colouring properties and they appear in plenty of constructions of cyclically 5-connected snarks. However, they can not be used in constructions of cyclically 6-connected snarks where we miss convenient multipoles. Therefore, we illustrate several methods for constructing multipoles with at least 6 dangling edges and useful colouring properties. These properties mostly include that for each colouring of a given multipole, flow through some of its connectors is zero or non-zero. Each of the following lemmas describes one construction of some multipole and its colouring properties.

**Lemma 5.12.** *Let $e$, $f$ be a pair of removable edges of a snark $S$ and $g$ another edge of $S$. Then for each colouring $\varphi$ of the $(2, 2, 2)$-pole $M_1(C_1, C_2, C_3) = S - \{e, f, g\}$, one has that $\varphi_\Sigma(C_3) \neq 0$.*

*Proof.* To the contrary, assume that $\varphi_\Sigma(C_3) = 0$ for some colouring $\varphi$ of the $(2,2,2)$-pole $M_1$. Then the $(2,2)$-pole $S - \{e, f\}$ is also colourable which contradicts the fact that the pair of the edges $\{e, f\}$ is removable. □

**Lemma 5.13.** *Let $e$, $f$, $g$ be three distinct edges of a snark $S$ such that each pair of them is removable in $S$. Ten the $(2,2,2)$-pole $M_2(C_1, C_2, C_3) = S - (e, f, g)$ is proper, in other words, for each colouring $\varphi$ of $M_2$, $\varphi_\Sigma(C_1) \neq 0$, $\varphi_\Sigma(C_2) \neq 0$ and $\varphi_\Sigma(C_3) \neq 0$.*

*Proof.* It is a corollary of Lemma 5.12. □

**Lemma 5.14.** *Let $e$, $f$ be a pair of removable edges of a snark $S$ and $v \in V(S)$. Then for each colouring $\varphi$ of the $(2,2,3)$-pole $M_3(C_1, C_2, C_3) = S - \{e, f\} - v$, one has that $\varphi_\Sigma(C_3) \neq 0$.*

*Proof.* Existence of a colouring $\varphi$ of $M_3$ such that $\varphi(C_3) = 0$ would imply an existence of a colouring of $S - \{e, f\}$. □

**Lemma 5.15.** *Let $uv$, $xy$ be a pair of removable edges of a snark $S$. Consider the $(3,3)$-pole $M_4((e_1, e_2, e_3), (f_1, f_2, f_3)) = S - \{u, x\}$ where the semiedges $e_1$, $f_1$ are incident with the vertices $v$, $y$, respectively. Then for each colouring $\varphi$ of $M_4$ one has that $\varphi(e_2) + \varphi(e_3) = 0$ or $\varphi(f_2) + \varphi(f_3) = 0$.*

*Proof.* Suppose the contrary. Then we can add to the multipole $M_4$ one vertex and connect it with the semiedges $e_2$, $e_3$ and similarly, add another vertex incident with the semiedges $f_2$, $f_3$. Afterwards, we get a $(2,2)$-pole isomorphic to $S - \{e, f\}$ which can be coloured by a simple extension of the colouring $\varphi$—a contradiction. □

**Lemma 5.16.** *Let $u$, $v$ be a pair of removable vertices of a snark $S$ and $w$ a neighbour of $u$. Then for each colouring $\varphi$ of the $(3,2,2)$-pole $M_5((e_1, e_2, e_3), (f_1, f_2), (g_1, g_2)) = S - v - [u, w]$, one has that $\varphi(g_1) + \varphi(g_2) = 0$.*

*Proof.* If we had $\varphi(g_1) + \varphi(g_2) \neq 0$, then the $(3,3)$-pole $S - \{u, v\}$ would be colourable by a extension of the colouring $\varphi$. □

**Lemma 5.17.** *Let $u$, $v$ be a pair of removable vertices of a snark $S$ and let $w$ be another vertex of $S$. Then for each colouring $\varphi$ of the $(3,3,3)$-pole $M_6(C_1, C_2, C_3) = S - \{u, v, w\}$ one has that $\varphi_\Sigma(C_3) \neq 0$.*

*Proof.* If we had that $\varphi_\Sigma(C_3) = 0$ for some colouring $\varphi$ of $M_6$, then we could extend the colouring $\varphi$ to a colouring of $S - \{u, v\}$. □

**Lemma 5.18.** *Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ be five odd $(2,2,2)$-poles (see Section 2.5). Then the multipole $K(C_2, C_3, C_4)$ (the connector $C_i$ contains the semiedges of the multipole $H_i$) depicted in Figure 5.2 is also an odd $(2,2,2)$-pole.*

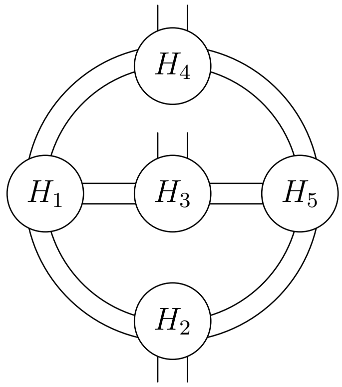



Figure 5.2

*Proof.* To the contrary, suppose that the $(2, 2, 2)$-pole $K$ is colourable by a colouring $\varphi$ such that $\varphi_\Sigma(C_i) \neq 0$ for each $i$. This implies $\varphi_\Sigma(C_2) = a$, $\varphi_\Sigma(C_3) = b$ and $\varphi_\Sigma(C_4) = c$ for $\{a, b, c\} = \mathbb{K}$. According to the properties of $H_4$, flow through one of its remaining two connectors has to be 0 and through the other $a$, so without loss of generality, let the flow between $H_4$ and $H_5$ is $a$. Therefore, flow between $H_5$ and $H_i$ for some $i \in \{2, 3\}$ is equal to $a$. However, the flow between $H_i$ and $H_1$ is zero by Parity Lemma. That is a contradiction with the colouring properties of $H_i$. □

This odd $(2, 2, 2)$-pole $K$ can be completed to a snark of order 34 by performing a junction with $V_4$ and this snark is described also in our bachelor thesis [19] in Class 34-6.

The described multipoles can be combined to create a snark in various ways. Since these multipoles have width at least 6 we can obtain snarks with girth 6 or cyclic connectivity 6 when we use appropriate snarks. We illustrate several methods of using aforementioned multipoles to construct snarks from $\mathcal{S}(5, 6)$ and $\mathcal{S}(6, 6)$ in the following examples.

**Example 3.** We start with a snark $S_{52}$ of the class NNN consisting of three negators from the Isaacs snark $J_5$ which has order 52. We choose a pair $\{e, f\}$ of its removable edges along with another edge $g$ of the snark $S_{52}$ and according to Lemma 5.12, we construct the $(2, 2, 2)$-pole $M_{52}((f_1, f_2), (f_3, f_4), (f_5, f_6)) = S_{52} - (e, f, g)$. Then, we take the strictly critical snark $S_{66}$ of order 66 depicted in Figure 3.6 and choose a pair $\{u, v\}$ of its non-adjacent removable vertices along with a neighbour $w$ of the vertex $u$. According to Lemma 5.16, we construct the $(3, 2, 2)$-pole $M_{63}((e_1, e_2, e_3), (e_4, e_5), (e_6, e_7)) = (S_{66} - v) - [u, w]$. If we perform a junction of the connectors $(e_6, e_7)$ and $(f_5, f_6)$, we get an uncolourable 9-pole $M_{115}$ of order 115. Such 9-pole can be completed to a snark or order 116 in many ways—one completion is depicted in Figure 5.3 and it has order 116 and girth 6. The links joining the two incompatible connectors $(e_6, e_7)$, $(f_5, f_6)$ are marked by the dashed line.

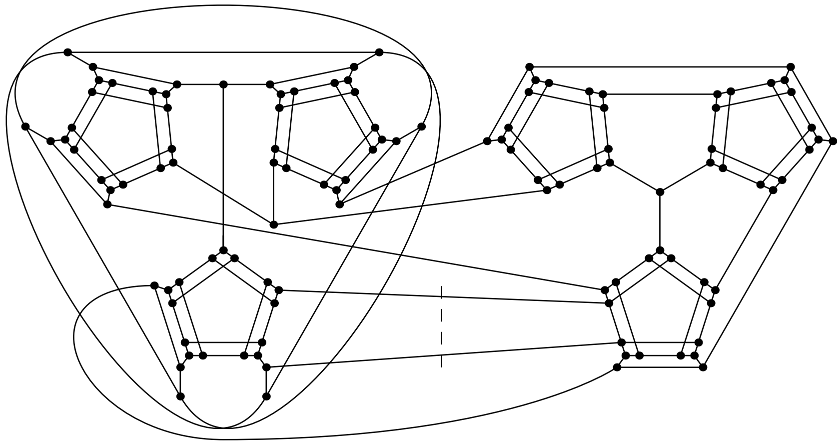


Figure 5.3: A snark of order 116 and girth 6

**Example 4.** For the second example, we take a cyclically 6-connected snark $S_{118}$ of order 118 constructed by Kochol in [12] and choose a pair of its removable edges $\{uv, xy\}$. Therefore, we construct a $(2,2,2)$-pole $M_{116}(C_1, C_2, C_3)$ according to Lemma 5.15—we remove the vertices $u$ and $x$ from the snark $S$ and group the accrued semiedges in three connectors in such a way that for each colouring $\varphi$ of $M_{116}$ one has $\varphi_\Sigma(C_1) = 0$ or $\varphi_\Sigma(C_2) = 0$. Additionally, we construct two copies $M_{118}(D_1, D_2, D_3)$ and $N_{118}(E_1, E_2, E_3)$ of $(2,2,2)$-pole $S_{118} - (ev, xy, f)$ where $f$ is another edge of $S_{118}$ non-incident with $uv$ and $xy$. After a disjoint union of the multipoles $M_{116}$, $M_{118}$ and $N_{118}$ and performing junctions of the connectors $C_1$ and $D_3$, $C_2$ and $E_3$, we get an uncolourable 10-pole which can be completed to a snark of order 352. A scheme of one such snark is shown in Figure 5.4. The computer verification showed that this snark is cyclically 6-connected.

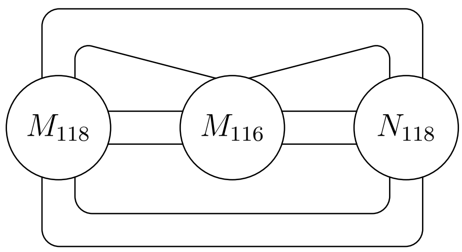



Figure 5.4: A scheme of a cyclically 6-connected snark of order 352

Lemma 5.13 seems very promising since every $(2,2,2)$-pole can be completed to a snark with only a 6-cycle. In our bachelor thesis [19], we constructed a proper $(2,2,2)$-pole of order 244 which is not acyclic and has no cuts of size 5 or smaller. In the following example, we construct such proper $(2,2,2)$-pole of order 118.

**Example 5.** The Kochol's snark $S_{118}$ contains three edges $e$, $f$, $g$ such that each pair of them is removable. Therefore, the $(2,2,2)$-pole $V = S - (e,f,g)$ is proper and $S_{124} = V * C_6'$ is a snark. We verified that the snark $S_{124}$ is cyclically 6-connected.

We do not know if these constructions can be used to construct smaller snarks. However, it is difficult to find small snarks suitable for this construction. Most of them either contain no pairs of removable edges and thereby no removable pairs of vertices (the Isaacs snarks, the double star snark) or too many cuts of size 5 or smaller.

# Conclusion

In our thesis, we explained the structure of strictly critical snarks of order up to 36. Using these results alongside the results of our bachelor's thesis [19] we constructed several infinite classes of critical or strictly critical snarks with cyclic connectivity 5 or 6.

To be able to prove criticality or bicriticality of snarks contained in the examined classes, we had to impose additional requirements on used construction blocks. Testing these requirements can be performed by a straightforward verification using a computer. The effectiveness of such tests is comparable with the effectiveness of tests of bicriticality since they test the colourability of some graph $O(n^2)$ times where $n$ is the order of the examined multipole. Despite that fact, it is still more efficient than to test the criticality of a resulted snark since it could have thrice as large an order as the used constructions blocks and determining whether a cubic graph is colourable is an NP-complete problem. Moreover, if we would like to generate a large number of critical snarks, we can prepare a set of admissible multipoles and connect them in all possible ways.

Additionally, we solved two problems proposed in [6]. In Theorem 3.3 we constructed infinitely many cyclically 6-connected strictly critical snarks solving Problem 6.3. Also, we solved Problem 5.7 by providing an example of bicritical snark which contains a pair of non-removable edges that is not essential.

In Chapter 5, we generated all non-trivial snarks with girth 6, cyclic connectivity at most 5 and order 40. We utilised the decomposition theorems of Goldberg [10] and Cameron et al. [18] to compose the desired snarks from two smaller ones. A considerable obstacle we had to overcome was to deal with trivial snarks which could appear in the decomposition of a cyclically 5-connected snark. We had to generate a list of all trivial 3-connected snarks with up to 30 vertices

The last chapter also describes several methods on how to obtain multipoles with width at least 6 and useful colouring properties. These multipoles can be used in constructions of snarks with girth 6 or cyclic connectivity 6 as we illustrated in three examples.

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