Druhá písomka z UDML, 15.12.2016 - riešenia

(Celý materiál je neoficiálny, ak tu mám chybu ja, vaše riešenie s rovnakou chybou **nie je** správne!)

1. Definujte formuly predikátovej logiky a podformuly.
   Vypíšte všetky podformuly pre formulu: $(\forall x)(\neg(x = y) \to (\exists y)((P(x,y) \vee \neg R(x,y,z)) \to (x+y=z)))$
   Označte tie, ktoré sú atomické. [2b]

- Definícia formuly:
  1. Nech P je n-árny predikátový symbol a $t_1, t_2, \ldots, t_n$ sú termy. Potom $P(t_1, t_2, \ldots t_n)$ je formula. Hovoríme jej tiež atomická formula.
  2. Nech A a B sú formuly, potom $(A \leftrightarrow B), (A \to B), (A \wedge B), (A \vee B), \neg A$ sú formuly.
  3. Nech A je formula a x je premenná, potom $(\forall x)A$ a $(\exists x)A$ sú formuly.
  4. Každá formula vznikne konečným použitím 1., 2. a 3.

- Definícia podformuly (prvá možnosť): Podformula formuly A je formula, ktoré je jej podreťazcom.

- Definícia podformuly (druhá možnosť):
  1. Ak A je atomická formula, potom je jej podformulou iba ona sama.
  2. Ak A je tvaru $B \leftrightarrow C, B \to C, B \wedge C$ alebo $B \vee C$, potom je podformulou ona sama a každá podformula formuly B alebo C.
  3. Ak A je tvaru $\neg B, (\forall x)B$ alebo $(\exists x)B$, potom je podformulou ona sama a každá podfromula formuly B.
  4. Žiadne ďalšie podformuly neexistujú.

- Podformuly formuly zo zadania:
  1. $x = y$
  2. $P(x,y)$
  3. $R(x,y,z)$
  4. $x + y = z$
  5. $\neg(x = y)$
  6. $\neg R(x,y,z)$
  7. $P(x,y) \vee \neg R(x,y,z)$
  8. $(P(x,y) \vee \neg R(x,y,z)) \to (x+y=z)$
  9. $(\exists y)((P(x,y) \vee \neg R(x,y,z)) \to (x+y=z))$
  10. $\neg(x=y) \to (\exists y)((P(x,y) \vee \neg R(x,y,z)) \to (x+y=z))$
  11. $(\forall x)(\neg(x=y) \to (\exists y)((P(x,y) \vee \neg R(x,y,z)) \to (x+y=z)))$

  Atomické podfromuly sú prvé štyri.

2 a) Vo FS PL dokážte prenexnú operáciu $(\forall x)(A \wedge B) \leftrightarrow ((\forall x)A \wedge B)$ kde x nie je voľná v B.
   Môžete použiť prvú, druhú aj tretiu prenexnú operáciu, tj. $(Qx)\neg A \leftrightarrow \neg(\bar{Q}x)A$,
   $(Qx)(B \to A) \leftrightarrow (B \to (Qx)A)$ a $(Qx)(B \to A) \leftrightarrow ((\bar{Q}x)B \to A)$.
   b) Preveďte nasledujúcu formulu do prenexného normálneho tvaru:
   $(\exists x)P(x) \wedge \neg(\forall y)[(\forall x)Q(x,y) \to (\forall x)(\exists y)(R(x,y) \to P(y))]$ [4b]

a) V tomto dôkaze budem prakticky stále používať vetu o ekvivalencií. To znamená, že budem meniť podformulu za jej ekvivalentnú.

$\vdash$	$(\exists x)(A \to \neg B) \leftrightarrow (\exists x)(A \to \neg B)$	(T1)
$\vdash$	$((\forall x)A \to \neg B) \leftrightarrow (\exists x)(A \to \neg B)$	(3. prenexná operácia (p.o.))
$\vdash$	$\neg(\exists x)(A \to \neg B) \leftrightarrow \neg((\forall x)A \to \neg B)$	(obmena)
$\vdash$	$(\forall x)\neg(A \to \neg B) \leftrightarrow \neg((\forall x)A \to \neg B)$	(1. p.o.)
$\vdash$	$(\forall x)(A \wedge B) \leftrightarrow ((\forall x)A \wedge B)$	$((A \wedge B) \leftrightarrow \neg(A \to \neg B))$

b)

$(\exists x)P(x) \wedge \neg(\forall y)[(\forall x)Q(x,y) \to (\forall x)(\exists y)(R(x,y) \to P(y))]$	
$(\exists x)P(x) \wedge \neg(\forall y)(\forall x)[(\forall x)Q(x,y) \to (\exists y)(R(x,y) \to P(y))]$	(2. p.o. na $(\forall x)$)
$(\exists x)P(x) \wedge \neg(\forall y)(\forall x)[(\forall x)Q(x,y) \to (\exists w)(R(x,w) \to P(w))]$	(premenovanie 2. y na w)
$(\exists x)P(x) \wedge \neg(\forall y)(\forall x)(\exists w)[(\forall x)Q(x,y) \to (R(x,w) \to P(w))]$	(2. p.o. na $(\forall w)$)
$(\exists x)P(x) \wedge \neg(\forall y)(\forall x)(\exists w)[(\forall u)Q(u,y) \to (R(x,w) \to P(w))]$	(premenovanie 3. x na u)
$(\exists x)P(x) \wedge \neg(\forall y)(\forall x)(\exists w)(\exists u)[Q(u,y) \to (R(x,w) \to P(w))]$	(3. p.o. na $(\forall u)$)
$(\exists x)P(x) \wedge (\exists y)(\exists x)(\forall w)(\forall u)\neg[Q(u,y) \to (R(x,w) \to P(w))]$	(4 x 1. p.o.)
$(\exists x)(P(x) \wedge (\exists y)(\exists x)(\forall w)(\forall u)\neg[Q(u,y) \to (R(x,w) \to P(w))])$	(5. p.o. na $(\exists x)$)
$(\exists x)(P(x) \wedge (\exists y)(\exists v)(\forall w)(\forall u)\neg[Q(u,y) \to (R(v,w) \to P(w))])$	(premenovanie 2. x na v)
$(\exists x)(\exists y)(\exists v)(\forall w)(\forall u)(P(x) \wedge \neg[Q(u,y) \to (R(v,w) \to P(w))])$	(4 x 4. p.o.)

3 Uvažujme Robinsonovu aritmetiku, tj. teóriu v predikátovej logike s rovnosťou so špeciálnymi axiómami:

(Q1)	$(\forall x)(\forall y)(S(x) = S(y) \to x = y)$	(Q4)	$(\forall x)(x + 0 = x)$
(Q2)	$(\forall x)\neg(S(x) = 0)$	(Q5)	$(\forall x)(\forall y)(x + S(y) = S(x + y))$
(Q3)	$(\forall x)(\neg(x = 0) \to (\exists y)(x = S(y)))$	(Q6)	$(\forall x)(x \cdot 0 = 0)$
		(Q7)	$(\forall x)(\forall y)(x \cdot S(y) = x \cdot y + x)$

Dokážte: (a) $0 \cdot S(0) = 0$, (b) $(\forall x)[(x = S(0)) \to (\forall y)(y + x = S(y))]$. [4b]

a) Za 1.5b:

1.	$\vdash$	$(\forall x)(\forall y)(x.S(y) = x.y + x)$	(Q7)
2.	$\vdash$	$0.S(0) = 0.0 + 0$	(2x(AŠ na 1. (x=0, y=0), MP))
3.	$\vdash$	$0.0 = 0$	(Q6, AŠ (x=0), MP)
4.	$\vdash$	$(\forall x)(x = x)$	(PG na R1)
5.	$\vdash$	$0 = 0$	(AŠ(x=0),MP)
6.	$\vdash$	$(\forall x_1)(\forall y_1)(\forall x_2)(\forall y_2)(x_1 = y_1 \to (x_2 = y_2 \to x_1 + x_2 = y_1 + y_2))$	(4xPG na R2)
7.	$\vdash$	$0.0 = 0 \to (0 = 0 \to 0.0 + 0 = 0 + 0)$	(4x(AŠ na 6. (x_1=0.0, y_1=0, x_2=0, y_2=$S(0)$), MP))
8.	$\vdash$	$0.0 + 0 = 0 + 0$	(2x MP 3. a 5. na 7.)
9.	$\vdash$	$0 + 0 = 0$	(Q4, AŠ (x=0), MP))
10.	$\vdash$	$0.S(0) = 0$	(2x tranzitivita = na 2., 8. a 9.)

b) Za 2.5b:

1.		$\vdash$	$(\forall x)(\forall z)(x + S(z) = S(x + z))$	(Q5)
2.		$\vdash$	$y + S(0) = S(y + 0)$	(2x(AŠ na 1. (x=y, z=0), MP))
3.		$\vdash$	$y + 0 = y$	(Q4, AŠ (x=y), MP)
4.		$\vdash$	$(\forall x)(\forall z)(x = z \to S(x) = S(z))$	(2x PG na R2)
5.		$\vdash$	$y + 0 = y \to S(y + 0) = S(y)$	(2x(AŠ na 4. (x=y+0,z=y), MP))
6.		$\vdash$	$S(y + 0) = S(y)$	(MP 3.,5.)
7.		$\vdash$	$y + S(0) = S(y)$	(tranzitivita = na 2. a 6.)
8.		$\vdash$	$(\forall x_1)(\forall y_1)(\forall x_2)(\forall y_2)(x_1 = y_1 \to (x_2 = y_2 \to x_1 + x_2 = y_1 + y_2))$	(4xPG na R2)
9.		$\vdash$	$y = y \to (x = S(0) \to y + x = y + S(0))$	(4x(AŠ (x_1=y, y_1=y, x_2=x, y_2=$S(0)$), MP))
10.		$\vdash$	$y = y$	(R1)
11.		$\vdash$	$x = S(0) \to y + x = y + S(0)$	(MP 9., 10.)
12.	$x = S(0)$	$\vdash$	$y + x = y + S(0)$	(VD)
13.	$x = S(0)$	$\vdash$	$y + x = S(y)$	(tranzitivita = na 12. a 7.)
14.	$x = S(0)$	$\vdash$	$(\forall y)(y + x = S(y))$	(PG na y, y nemá na ľavej strane voľný výskyt)
15.		$\vdash$	$x = S(0) \to (\forall y)(y + x = S(y))$	(VD)
16.		$\vdash$	$(\forall x)(x = S(0) \to (\forall y)(y + x = S(y)))$	(PG)

4 Pre každú dvojicu teória T, formula A dokážte $T \vdash A$ vo FS PL, alebo nájdite model $\mathcal{M} \models T$ taký, že $\mathcal{M} \not\models A$.

(a) $T = \{(\forall x)(\forall y)(x = y), (\exists x)(\exists y)\neg(x = y)\}$, $A\colon (\forall x)(P(x) \wedge \neg P(x))$

(b) $T = \{(\forall x)(Q(x) \vee R(x) \to P(x)), (\exists x)(P(x) \wedge \neg Q(x)), (\forall x)(Q(x) \to R(x)\}$, $A\colon (\forall x)(P(x) \wedge R(x))$

(c) $T = \{(\forall x)(\exists y)(x + y = 0), (\forall x)(x + 0 = x)\}$, $A\colon (\exists x)(\exists y)(y + y = x)$

(d) $T = \{(\forall x)(\exists y)(x + y = 0), (\forall x)(x + 0 = x)\}$, $A\colon (\forall x)(0 + x = x)$

(e) $T = \{(\forall x)\neg(x < x), (\forall x)(\forall y)[(x < y) \to \neg(y < x)], (\forall x)(\forall y)(\forall z)[(x < y) \to ((y < z) \to (x < z))]\}$, $A\colon (\forall x)[\neg(x = c) \to (x < c)] \to \neg(\exists x)(c < x)$ [5b]

(a) T je sporná teória. Taká teória nemá model, ale zato sa z nej dajú dokázať všetky formuly. Jeden z dôkazov našej formuly môže byť:

1.	T	$\vdash$	$(\forall x)(\forall y)(x = y)$	(predpoklad)
2.	T	$\vdash$	$(\exists x)(\exists y)\neg(x = y)$	(predpoklad)
3.	T	$\vdash$	$\neg(\forall x)(\forall y)(x = y)$	(prepis $\exists$ + veta o ekvivalencií)
4.	T	$\vdash$	$\neg(\forall x)(\forall y)(x = y) \to ((\forall x)(\forall y)(x = y) \to (\forall x)(P(x) \wedge \neg P(x)))$	(T2)
5.	T	$\vdash$	$(\forall x)(P(x) \wedge \neg P(x))$	(2x MP 1. a 3. na 4.)

(b) $T \not\vdash A$. Asi najjednoduchší model T, ktorý nie je modelom A vyzerá takto:

$$\mathcal{M}\colon \mathcal{U} = \{0\}, \quad P_{\mathcal{M}} = \{0\}, \quad Q_{\mathcal{M}} = \emptyset, \quad R_{\mathcal{M}} = \emptyset$$

(c) $T \vdash A$:

1.	T	$\vdash$	$(\forall x)(x + 0 = x)$	(predpoklad)
2.	T	$\vdash$	$(\forall x)(x + 0 = x) \to (0 + 0 = 0)$	(AŠ x=0)
3.	T	$\vdash$	$0 + 0 = 0$	(MP 1., 2.))
4.	T	$\vdash$	$(0 + 0 = 0) \to (\exists y)(y + y = 0)$	(duálna AŠ)
5.	T	$\vdash$	$(\exists y)(y + y = 0)$	(MP 3., 4.)
6.	T	$\vdash$	$(\exists y)(y + y = 0) \to (\exists x)(\exists y)(y + y = x)$	(duálna AŠ)
7.	T	$\vdash$	$(\exists x)(\exists y)(y + y = x)$	(MP 5., 6.)

(d) $T \not\vdash A$. Vyplýva to z toho, že nemáme zabezpečené, že + je komutatívne. Stačí zvoliť napríklad, takúto relačnú štruktoru $\mathcal{M}$:

$$\mathcal{M}\colon \mathcal{U} = \mathbb{Z}, \quad 0_{\mathcal{M}} = 0, \quad +_{\mathcal{M}} = -$$

Čiže plus realizujeme ako operáciu odčítanie. Táto relačná štruktúra je modelom T a nie je modelom A.

(e) $T \vdash A$:

1.	$T, (\forall x)(\neg(x = c) \to (x < c))$	$\vdash$	$(\forall x)(\neg(x = c) \to (x < c))$	(predpoklad)
2.	$T, (\forall x)(\neg(x = c) \to (x < c))$	$\vdash$	$\neg(x = c) \to (x < c)$	(AŠ (x=x), MP)
3.	$T, (\forall x)(\neg(x = c) \to (x < c)), \neg(x = c)$	$\vdash$	$(x < c)$	(VD)
4.	T	$\vdash$	$(x < c) \to \neg(c < x)$	(predp., 2x(AŠ (x=x, y=c), MP))
5.	$T, (\forall x)(\neg(x = c) \to (x < c)), \neg(x = c)$	$\vdash$	$\neg(c < x)$	(MP 3., 4.)
6.	$T, (\forall x)(\neg(x = c) \to (x < c)), (x = c)$	$\vdash$	$(x = c)$	(MP 3., 4.)
7.		$\vdash$	$(\forall x)(\forall y)(x = y \to y = x)$	(2xPG na symetriu =)
8.		$\vdash$	$(x = c) \to (c = x)$	(2x(AŠ (x=x, y=c), MP))
9.	$T, (\forall x)(\neg(x = c) \to (x < c)), (x = c)$	$\vdash$	$(c = x)$	(MP 6., 8.)
10.		$\vdash$	$x = x$	(R1)
11.		$\vdash$	$c = x \to (x = x \to (c < x \to x < x))$	(inštancia R3)
12.	$T, (\forall x)(\neg(x = c) \to (x < c)), (x = c)$	$\vdash$	$c < x \to x < x$	(2xMP 9. a 10. na 11.)
13.	$T, (\forall x)(\neg(x = c) \to (x < c)), (x = c)$	$\vdash$	$\neg(x < x) \to \neg(c < x)$	(obmena 12.)
14.	T	$\vdash$	$\neg(x < x)$	(predpoklad, AŠ (x=x), MP)
15.	$T, (\forall x)(\neg(x = c) \to (x < c)), (x = c)$	$\vdash$	$\neg(c < x)$	(MP 14., 13.)
16.	$T, (\forall x)(\neg(x = c) \to (x < c))$	$\vdash$	$\neg(c < x)$	(lema o neutr. formule 5., 15.)
17.	$T, (\forall x)(\neg(x = c) \to (x < c))$	$\vdash$	$(\forall x)\neg(c < x)$	(PG, x nemá naľavo voľný výskyt)
18.	$T, (\forall x)(\neg(x = c) \to (x < c))$	$\vdash$	$\neg(\exists x)(c < x)$	(1. p.o., MP)
19.	T	$\vdash$	$(\forall x)(\neg(x = c) \to (x < c)) \to \neg(\exists x)(c < x)$	(VD)