Algorithms and Data Structures for Mathematicians

Lecture 3: Basic Data Structures

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Dynamic Sets and Data Structures

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- Elements might not be implemented as objects, but we shall always think of them as pointers (not values)

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- Examples: arrays, linked lists, heaps, trees, ...

Dictionaries

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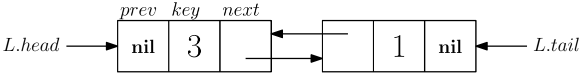
Dictionaries via (Doubly) Linked Lists



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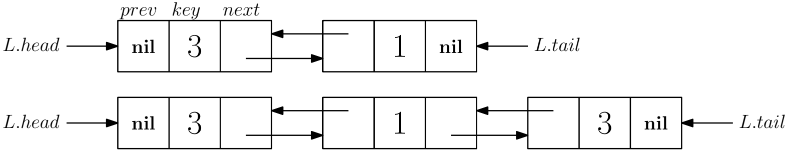

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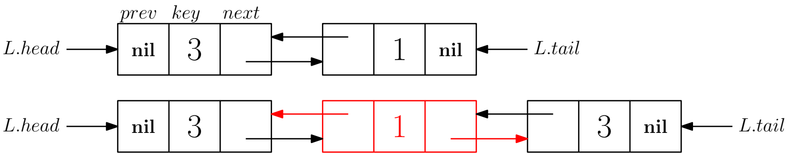

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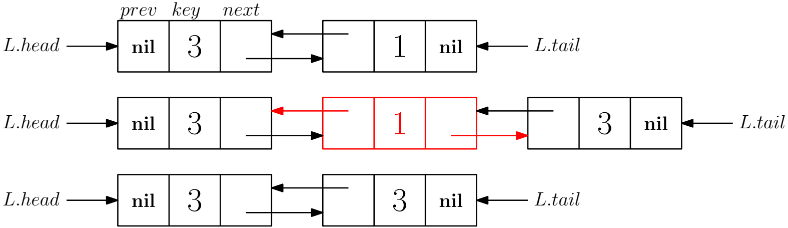

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$\text{LISTINSERT}(L, x)$:

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LISTINSERT(L, x):

Input : A linked list L; x not in L

Behaviour: Inserts x to L

```
x.prev ← L.tail;
x.next ← nil;
if L.tail ≠ nil then
 |  L.tail.next ← x
end
else
 |  L.head ← x
end
L.tail ← x;
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- Time complexity: $\Theta(1)$

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else
  $L.head \leftarrow x.next$
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if $x.next \neq$ **nil then**
  $x.next.prev \leftarrow x.prev$
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  $L.tail \leftarrow x.prev$
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Input : A linked list L; a key k
Output: First x in L such that $x.key = k$ or **nil** if there is no such x

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- Worst-case time complexity: $\Theta(n)$, where n is the number of elements in L

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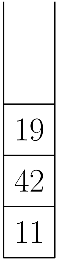

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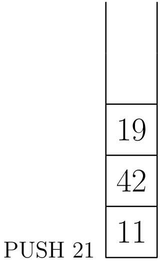

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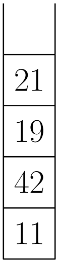

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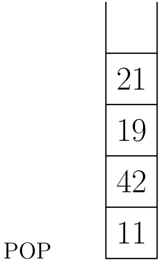

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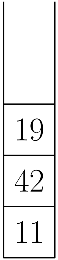

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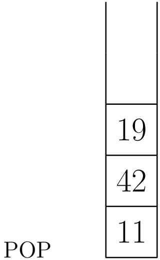

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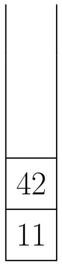

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POP(S):

Input : A stack S
Output: An element x on the top of the stack, which is removed from S

```
if S.num = 0 then
 |  error(underflow)
end
else
 |  S.num ← S.num − 1;
 |  return S.a[S.num + 1];
end
```

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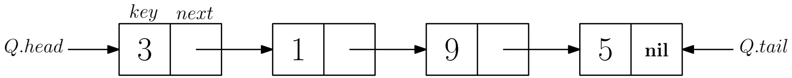
Queues via Singly Linked Lists



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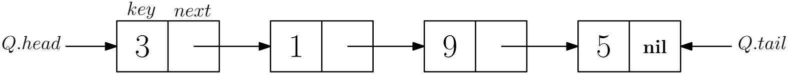

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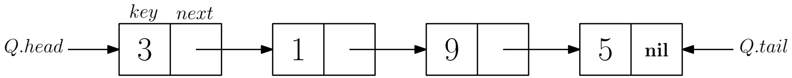

ENQUEUE(Q, x):

Input : A queue Q; x not in Q
Behaviour: Inserts x to Q

$x.next \leftarrow$ **nil**;
if $Q.tail \neq$ **nil then**
| $Q.tail.next \leftarrow x$
end
else
| $Q.head \leftarrow x$
end
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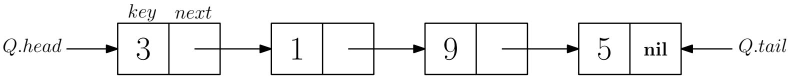
Queues via Singly Linked Lists



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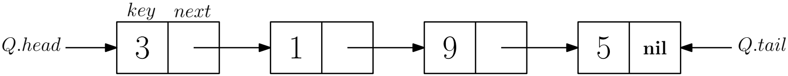

DEQUEUE(Q):

Input : A queue Q

Output: The element x inserted to Q the longest time ago, which is removed from Q

$x \leftarrow Q.head$;
if $x \neq$ **nil then**
| $Q.head \leftarrow x.next$
end
if $Q.head =$ **nil then**
| $Q.tail \leftarrow$ **nil**
end
return x;

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- $\mathrm{MAX}(Q)$: returns some x in Q with maximal $x.key$

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- $\mathrm{INSERT}(Q, x)$: inserts x into Q
- $\mathrm{MAX}(Q)$: returns some x in Q with maximal $x.key$
- $\mathrm{EXTRACTMAX}(Q)$: removes and returns some x in Q with maximal $x.key$

Priority Queues

Dynamic sets supporting the following operations (assuming that each x has an attribute $x.key$ from a totally ordered set $(S, \preceq)$):

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We shall realise priority queues via data structures called max-heaps

Max-Heaps

- "Nearly complete" binary trees

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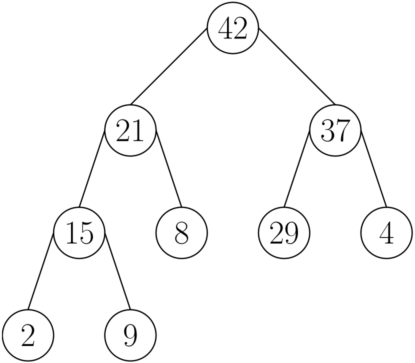

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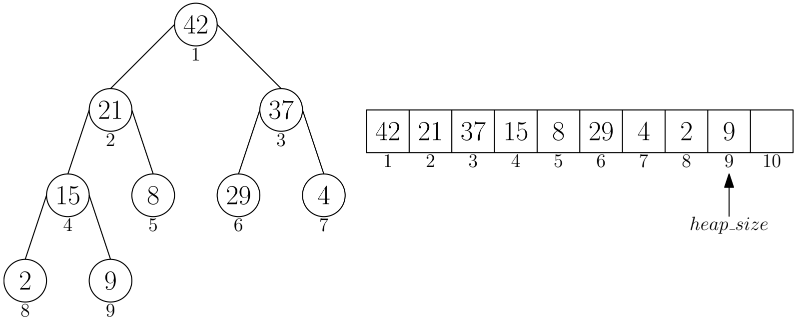

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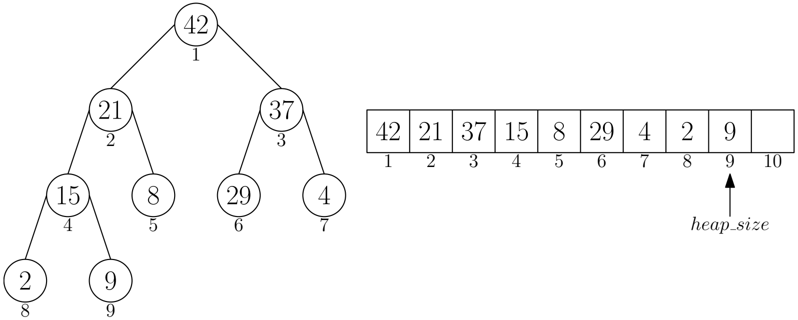

PARENT(i):

Max-Heaps

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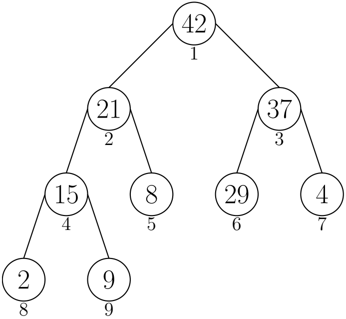

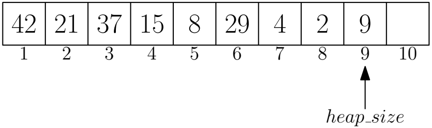

PARENT(i):

return $\lfloor i/2 \rfloor$;

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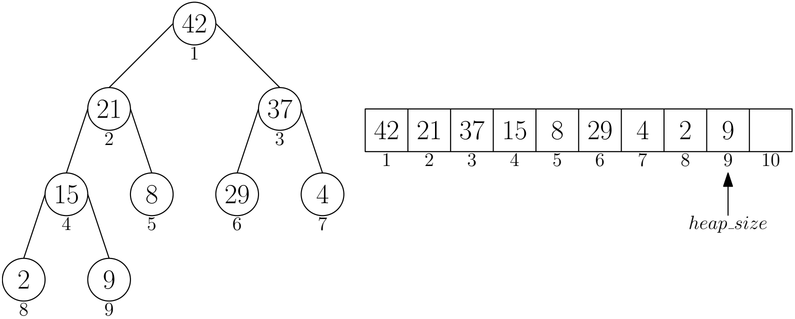

PARENT(i):

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LEFT(i):

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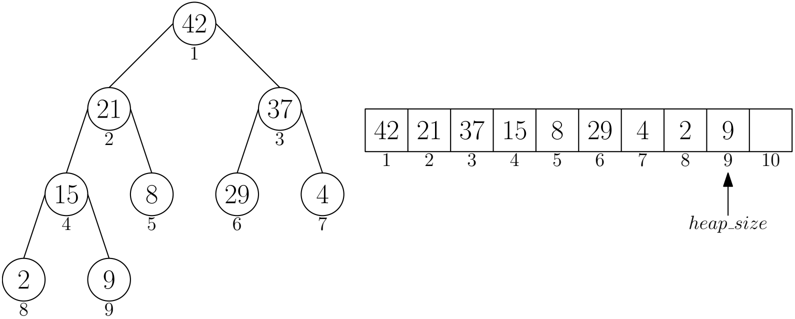

PARENT(i):

return $\lfloor i/2 \rfloor$;

LEFT(i):

return $2i$;

Max-Heaps

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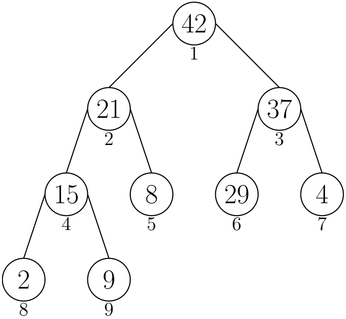

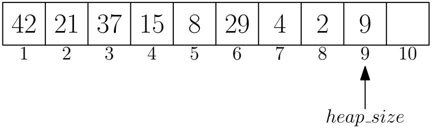

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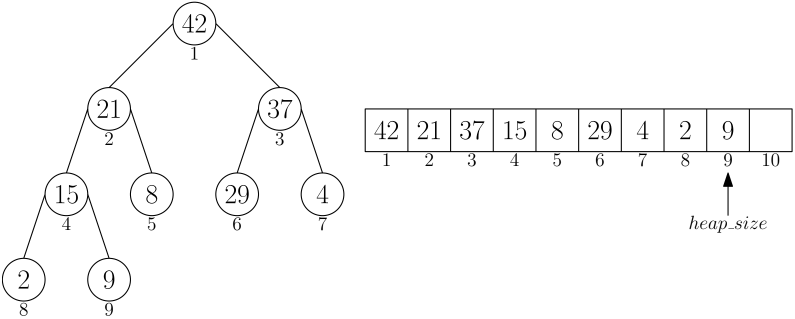

PARENT(i):

return $\lfloor i/2 \rfloor$;

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return $2i$;

RIGHT(i):

return $2i + 1$;

Max-Heaps

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- Assumption: both children of the i-th node are roots of valid heaps
- The i-th node can have a smaller key than some of its children
- The operation $\text{HEAPIFY}(H, i)$ transforms the subtree rooted at i into a valid heap

Max-Heaps

HEAPIFY(H, i):

Max-Heaps

$\text{HEAPIFY}(H, i)$:

$max \leftarrow i$;
if $\text{LEFT}(i) \leq H.heap\_size$ *and* $H[\text{LEFT}(i)].key \succ H[max].key$ **then**
  $max \leftarrow \text{LEFT}(i)$
end
if $\text{RIGHT}(i) \leq H.heap\_size$ *and* $H[\text{RIGHT}(i)].key \succ H[max].key$ **then**
  $max \leftarrow \text{RIGHT}(i)$
end
if $max \neq i$ **then**
  $H[i] \leftrightarrow H[max]$;
  $\text{HEAPIFY}(H, max)$;
end

Max-Heaps

$\text{HEAPIFY}(H, i)$:

```
max ← i;
if LEFT(i) ≤ H.heap_size and H[LEFT(i)].key ≻ H[max].key then
  | max ← LEFT(i)
end
if RIGHT(i) ≤ H.heap_size and H[RIGHT(i)].key ≻ H[max].key then
  | max ← RIGHT(i)
end
if max ≠ i then
  | H[i] ↔ H[max];
  | HEAPIFY(H, max);
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```

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- Worst-case over all nodes: $\Theta(\log n)$

Priority Queues via Max-Heaps

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EXTRACTMAX(H):

```
if H.heap_size = 0 then
    error(underflow)
end
else
    max ← H[1];
    H[1] ← H[H.heap_size];
    H.heap_size ← H.heap_size − 1;
    HEAPIFY(H, 1);
    return max;
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Priority Queues via Max-Heaps

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Priority Queues via Max-Heaps

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if k ≺ H[i].key then
 |  error(too small key)
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else
 |  H[i].key ← k;
 |  while i > 1 and H[PARENT(i)].key ≺ H[i].key do
 |   |  H[i] ↔ H[PARENT(i)];
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$k \leftarrow x.key$;
$x.key \leftarrow \perp$;
$H.heap\_size \leftarrow H.heap\_size + 1$;
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- Hence, $T(n) = O(n)$, implying that $T(n) = \Theta(n)$

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- As a result: $T(n) = \Theta(n \log n)$