Algorithms and Data Structures for Mathematicians

Lecture 4: Trees

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Binary Trees and Their Representation

From the last lecture:

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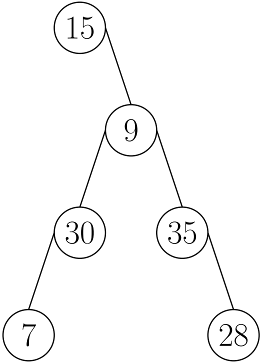
- We shall use an approach similar to linked lists instead
- Binary trees will simply be "branching" linked lists
- Generalisation to k-ary trees is straightforward

Binary Trees and Their Representation



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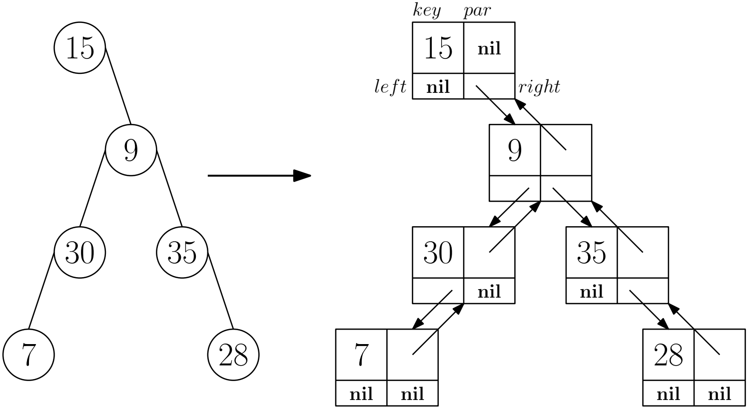

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These can be used to realise, e.g., dictionaries or priority queues

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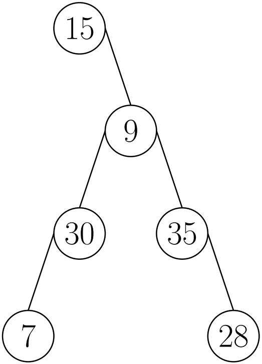

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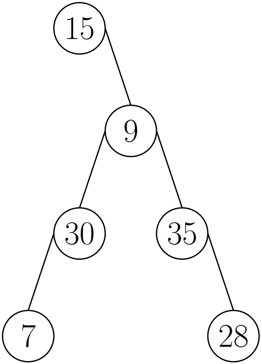

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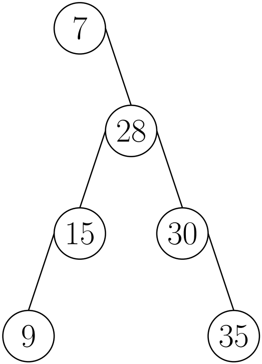

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$\text{BSTSEARCH}(x, k)$:

if $x =$ **nil** *or* $k = x.key$ **then return** x;
if $x \neq$ **nil** *and* $k \prec x.key$ **then return** $\text{BSTSEARCH}(x.left, k)$;
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if x.right ≠ nil then
|   return BSTMIN(x.right)
end
else
|   y ← y.par;
|   while y ≠ nil and x = y.right do
|   |   x = y;
|   |   y = y.par;
|   end
|   return y;
end
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- Predecessors can be found in a symmetric way

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par ← nil;
y ← T.root;
while y ≠ nil do
    par ← y;
    if x.key ≺ par.key then y ← par.left else y ← par.right;
end
x.par ← par;
if par = nil then T.root ← x
else
    if x.key ≺ par.key then par.left ← x
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  - If y is the right child of x, then replace x by y
  - Otherwise replace y by its right child and then x by y
- Replacements shall be done via $\text{BSTTRANSPLANT}(T, u, v)$

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else
  if $u = u.par.left$ **then** $u.par.left \leftarrow v$;
  if $u = u.par.right$ **then** $u.par.right \leftarrow v$;
end
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- Time complexity: $\Theta(1)$

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|   BSTTRANSPLANT(T, x, x.right)
else if x.right = nil then
|   BSTTRANSPLANT(T, x, x.left)
else
|   y ← BSTMIN(x.right);
|   if y.par ≠ x then
|   |   BSTTRANSPLANT(T, y, y.right);
|   |   y.right ← x.right;
|   |   y.right.par ← y;
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Balanced Binary Search Trees

For most operations on binary search trees:

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Elementary transformations retaining the binary search tree property

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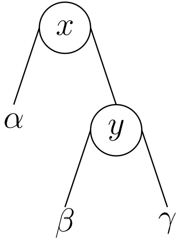
LEFTROTATE(x) :



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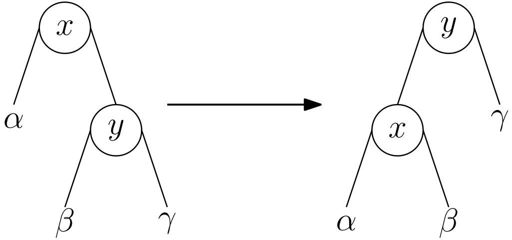

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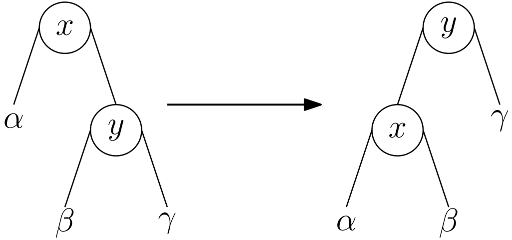

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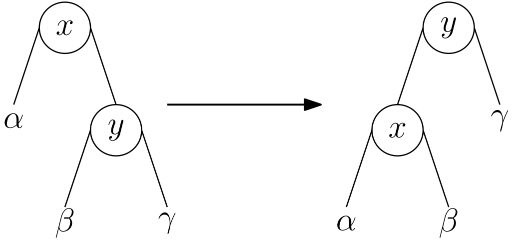

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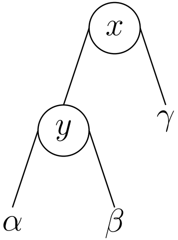

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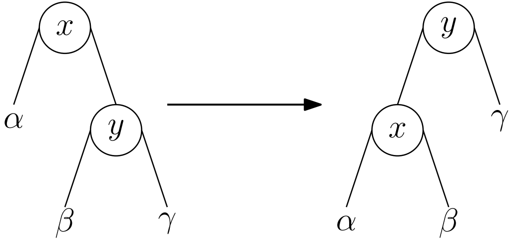

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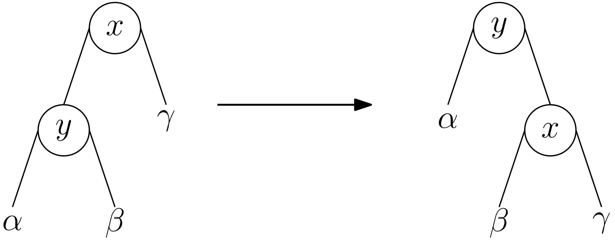

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$y \leftarrow x.right$;
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if $x.right \neq$ **nil then** $x.right.par \leftarrow x$;
$y.par \leftarrow x.par$;
if $y.par =$ **nil then** $T.root \leftarrow y$;
else
    if $x = y.par.left$ **then** $y.par.left \leftarrow y$;
    if $x = y.par.right$ **then** $y.par.right \leftarrow y$;
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- Time complexity: $\Theta(1)$

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```
y ← x.right;
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if x.right ≠ nil then x.right.par ← x;
y.par ← x.par;
if y.par = nil then T.root ← y;
else
    if x = y.par.left then y.par.left ← y;
    if x = y.par.right then y.par.right ← y;
end
y.left ← x;
x.par ← y;
```

- Time complexity: $\Theta(1)$
- RIGHTROTATE(T, x) can be done symmetrically

Red-Black Trees

Binary search trees such that:

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Binary search trees such that:

- Each node has an additional attribute – a colour

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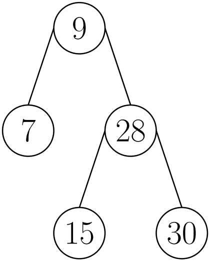
- There are not so many red nodes...

Red-Black Trees



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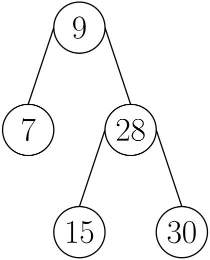

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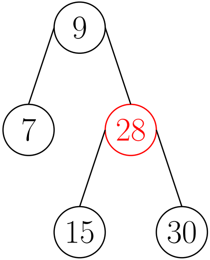

A red-black tree

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Lemma

Let T be a red-black tree with n nodes. Then the height of each node in T is at most $\Theta(\log n)$.

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- A tree with n nodes is of height $\Theta(\log n)$

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See Cormen et al. for details

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return $a_l \cdot \langle x \rangle \cdot a_r$;

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