Algorithms and Data Structures for Mathematicians

Lecture 5: Sorting

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Sorting Algorithms Covered So Far

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Conquer the subproblems by recursively applying the procedure to $\langle a[f], \ldots, a[m-1] \rangle$ and $\langle a[m+1], \ldots, a[l] \rangle$
(simply return for arrays with 1 or 0 elements)

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Combine phase is not needed here

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else

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m ← DIVIDE(a, f, l);
QUICKSORT(a, f, m − 1);
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DIVIDE(a, f, l):
piv ← a[l]; j ← f;
for i ← f to l − 1 do
    if a[i] ≼ piv then
        a[j] ↔ a[i];
        j ← j + 1;
    end
end
a[j] ↔ a[l];
return j;
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- For worst-case complexity, this implies $T(n) = \Omega(n^2)$

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  - $T(n) \leq c_2(n-1)^2 + 2c_2 + c_1 n = c_2 n^2 - 2c_2 n + c_2 + 2c_2 + c_1 n \leq c_2(n^2+1)$

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- Quick sort is particularly well suited for practical implementation (cache efficiency, etc.)

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RDIVIDE(a, f, l):

$index \leftarrow \text{RANDOM}(f, l)$;
$a[index] \leftrightarrow a[l]$;
return $\text{DIVIDE}(a, f, l)$;

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$$E\left[X\right] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E\left[X_{i,j}\right] =$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\left[s_i \text{ is compared with } s_j\right]$$

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$$
\begin{aligned}
\Pr[s_i \text{ is compared with } s_j] &= \Pr[s_i \text{ is a first pivot chosen from } S[i,j]] + \\
&+ \Pr[s_j \text{ is a first pivot chosen from } S[i,j]] = \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}
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- Hence, the expected time complexity is $\Theta(n \log n)$

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Sorting by Comparison

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- For instance merge sort and heap sort are thus optimal (among comparison sorts)

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Example of a decision tree for $n = 3$:

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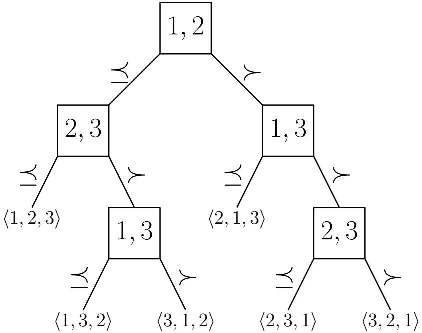

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- This is just an example, there are some other "similar" algorithms as well

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Create arrays $b = \langle b[1], \ldots, b[n] \rangle$ and $c = \langle c[0], \ldots, c[k] \rangle$;
for $i \leftarrow 0$ **to** k **do** $c[i] \leftarrow 0$;
for $j \leftarrow 1$ **to** n **do** $c[a[j]] \leftarrow c[a[j]] + 1$;
for $i \leftarrow 0$ **to** k **do** $c[i] \leftarrow c[i] + c[i-1]$;
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- Time complexity: $\Theta(n + k)$