# Reversible Weighted Automata over Finite Rings and Monoids with Commuting Idempotents*

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**Abstract.** Reversible weighted automata are introduced and considered in a specific setting where the weights are taken from a nontrivial locally finite commutative ring such as a finite field. It is shown that the supports of series realised by such automata are precisely the rational languages such that the idempotents in their syntactic monoids commute. In particular, this is true for reversible weighted automata over the finite field $\mathbb{F}_2$, where the realised series can be directly identified with such languages. A new automata-theoretic characterisation is thus obtained for the variety of rational languages corresponding to the pseudovariety of finite monoids **ECom**, which also forms the Boolean closure of the reversible languages in the sense of J.-É. Pin. The problem of determining whether a rational series over a locally finite commutative ring can be realised by a reversible weighted automaton is decidable as a consequence.

**Keywords:** Weighted automaton · Reversible automaton · Finite field · Variety of languages · Decidability

## 1 Introduction

This article embarks upon the study of *reversibility* in the context of *weighted automata*. More precisely, we focus here on a very special case of weighted automata over *finite commutative rings*, including in particular the two-element Galois field $\mathbb{F}_2$. Strong connections to language theory turn out to arise.

The study of the concept of reversibility in computing goes back to the seminal work of R. Landauer [34]: according to his fundamental thermodynamical principle, any loss of information that takes place during a computation necessarily leads to some minimal amount of heat dissipation. This observation led C. H. Bennett [11] to consider logical reversibility of computations, and in particular the *reversible Turing machines*, in which any configuration has at most one preceding and at most one successor configuration. Similar ideas gradually inspired the development of an entire field of reversible computing.

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Starting by D. Angluin [3] defining – in the context of language inference – what is now usually called *bideterministic finite automata* [43, 51–53], many different variants of *reversible finite automata* have been introduced and extensively studied throughout the literature [2, 18, 19, 21–23, 35, 43, 45, 46], while some generalisations of the usual concepts of reversibility in finite automata have been considered as well [5, 6, 17, 20]. What all these models have in common is that unlike in the case of reversible Turing machines, reversibility turns out to be a real restriction for finite automata and leads to a loss in their expressivity. Hence the study of the corresponding classes of languages has been a substantial part of the aforementioned efforts.

The notion of a *reversible finite automaton* considered in this article is the one of J.-É. Pin [43] – in other words, a finite automaton is *reversible* if its transition relation is both deterministic and codeterministic; however, the automaton is allowed to have more than one initial as well as more than one final state. Thus, using the terminology usual in weighted automata theory, we may say that a reversible finite automaton is a nondeterministic finite automaton that is finitely sequential [8, 31, 33, 39–41] (i.e., "deterministic" with possibly more than one initial state) and its transpose is finitely sequential as well. This definition is not only very natural, but also leads to a well-behaved class of *reversible languages*, which forms a *positive variety* [43]; the corresponding pseudovariety of ordered monoids can be captured by the pseudoinequalities $x^\omega y^\omega = y^\omega x^\omega$ and $x^\omega \leq 1$, which also give rise to a natural characterisation of reversible languages in terms of forbidden configurations in their minimal deterministic automata [43, 27, 28]. Reversible finite automata with at most one initial and at most one final state are precisely the *bideterministic finite automata* [43, 51–53].

In this article, we lift the notion of reversibility in the sense of J.-É. Pin [43] to the setting of *weighted finite automata* – i.e., finite automata with transitions carrying weights taken from some algebra such as a semiring, encompassing a necessary shift from recognising languages towards realisation of *formal power series* in several noncommuting variables [12, 15, 16, 47, 48]. While the very origins of weighted automata theory go back already to the seminal article of M.-P. Schützenberger [49], the field has experienced a wave of renewed interest during the last decades, and still represents a very active area of study [15, 16]. It is thus relatively surprising that reversible weighted automata have attracted almost no attention so far, the only exception being the recent research on bideterministic weighted automata [29, 32].

The study of reversible weighted automata would not only be natural in view of the previous research on reversibility in automata theory, but it can also be motivated by the study of certain *decision problems* for weighted automata. The *determinisability* problem, in which one asks about the existence of a deterministic – or sequential [36] – equivalent of a given weighted automaton, attracted significant attention in recent years [9, 10, 24, 30]. While the problem was proved to be decidable in important settings such as over fields [9] and some complexity bounds have been obtained as well [10, 24], one still has no efficient algorithms for deciding the determinisability problem for sufficiently general classes of weighted automata. On the other hand, it has been shown that one can do better when deciding the existence of a *bideterministic* equivalent for a given weighted automaton [29, 32]. This indicates that it might make sense to also look at some other restrictions of weighted automata related to determinism, and in particular at the *reversible weighted automata*, which represent both a natural generalisation of bideterministic weighted automata, as well as a class of finitely sequential automata that is in general incomparable with the deterministic weighted automata when it comes to expressivity.

The aim of this article is to initiate a systematic research on reversible weighted automata, while we mostly focus here on a very special case, in which the weights are from a *locally finite commutative ring*. This includes the case of weighted automata and formal power series over *finite fields*, and in particular over the *two-element field* $\mathbb{F}_2$. The latter setting gives rise to a natural way of describing languages, as any rational series over $\mathbb{F}_2$ is at the same time a characteristic series of some rational language – in fact, the series over $\mathbb{F}_2$ correspond to what has been studied as *formal languages over* GF(2) by E. Bakinova et al. [7, 38]; the idea of describing languages using weighted automata over fields also appears in the study of *image-binary automata* of S. Kiefer and C. Widdershoven [25, 26]. In a similar spirit, a weighted automaton over a *finite* or *locally finite* (semi)ring gives rise to a *rational* language by taking the support of its behaviour, i.e., the language of all words with a nonzero coefficient in the realised series.

We show that regardless of a *nontrivial locally finite commutative ring* $R$ considered, the languages described by the *reversible weighted automata* over $R$ in this way always form the same class, namely the variety of languages corresponding to the pseudovariety **ECom** of all finite monoids with commuting idempotents. This is at the same time the Boolean closure of the positive variety of all reversible languages [43], i.e., the variety generated by such languages.

Note that the result applies in particular to reversible weighted automata over $\mathbb{F}_2$ – the languages described by such automata are precisely the languages from the aforementioned variety. This means that interpreting a reversible finite automaton as a reversible weighted automaton over $\mathbb{F}_2$ leads to an increase in the expressive power of the model and to better closure properties of the corresponding class of languages. At the same time, a new automata-theoretic characterisation of the variety of languages corresponding to **ECom** is obtained.

We also characterise the *series* realised by the reversible weighted automata over nontrivial *finite* commutative rings. This characterisation implies, together with effective decidability of membership of a monoid in **ECom**, the existence of an algorithm for deciding whether a weighted automaton over an effective *locally finite* commutative ring $R$ admits a reversible equivalent over $R$ or not.

## 2 Preliminaries

We write $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ for the sets of all *nonnegative* integers, integers, and rational numbers, $\mathbb{B}$ for the Boolean domain $\mathbb{B} = \{0, 1\}$, and $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{N}$. Alphabets are finite and nonempty, $\varepsilon$ denotes the empty word.

A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$, or simply $S$, such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, $\cdot$ distributes over $+$ from both sides, and $0$ is a zero in $(S, \cdot, 1)$. A *subsemiring* of $(S, +, \cdot, 0, 1)$ is a semiring $(T, +_T, \cdot_T, 0, 1)$, where $T \subseteq S$ and $+_T, \cdot_T$ are the restrictions of $+$ and $\cdot$ to $T$. The *subsemiring of $S$ generated by $G \subseteq S$* is the smallest subsemiring $\langle G \rangle$ of $S$ such that $G \subseteq \langle G \rangle$. A semiring $(S, +, \cdot, 0, 1)$ is *commutative* when $\cdot$ is, *finite* when $S$ is, *finitely generated* when $S = \langle G \rangle$ for some finite $G \subseteq S$, and *locally finite* when all finitely generated subsemirings of $S$ are finite. A semiring $(S, +, \cdot, 0, 1)$ is *nontrivial* if $S$ contains at least two elements, that is, if $0 \neq 1$. An important example of a semiring is the *Boolean semiring* $(\mathbb{B}, \vee, \wedge, 0, 1)$.

A *ring* (with unity) is a semiring $(R, +, \cdot, 0, 1)$ such that $(R, +, 0)$ is an abelian group. A *field* is a nontrivial commutative ring $(\mathbb{F}, +, \cdot, 0, 1)$ such that $(\mathbb{F} \backslash \{0\}, \cdot, 1)$ is an abelian group. The finite field with two elements is denoted by $\mathbb{F}_2$.

The *characteristic* of a ring $R$ is, in case it exists, the smallest $n \in \mathbb{N} \setminus \{0\}$ such that $\sum_{k=1}^{n} 1 = 0$ in $R$ – or zero when there is no such $n$.

The reader can consult [12, 15, 16, 47, 48] for the basics on formal power series in several noncommuting variables and weighted automata. A *formal power series* over a semiring $S$ and alphabet $\Sigma$ is a mapping $r \colon \Sigma^* \to S$; the value of $r$ upon $w \in \Sigma^*$ is denoted by $(r, w)$ and called the *coefficient* of $r$ at $w$, while one writes $r = \sum_{w \in \Sigma^*} (r, w)\, w$. The set of all series over $S$ and $\Sigma$ is denoted by $S\langle\langle \Sigma^* \rangle\rangle$. Given $r, s \in S\langle\langle \Sigma^* \rangle\rangle$, we define $r + s$ by $(r + s, w) = (r, w) + (s, w)$ and $r \cdot s$ by $(r \cdot s, w) = \sum_{u,v \in \Sigma^*, uv = w} (r, u)(s, v)$ for all $w \in \Sigma^*$. Each $a \in S$ is identified with $r_a \in S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_a, \varepsilon) = a$ and $(r_a, w) = 0$ for all $w \in \Sigma^+$; the left or right multiplication by $a$ thus corresponds to left or right *scalar multiplication*. Similarly, each $w \in \Sigma^*$ is identified with $r_w \in S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_w, w) = 1$ and $(r_w, x) = 0$ for all $x \in \Sigma^* \setminus \{w\}$.

The *support* of $r \in S\langle\langle \Sigma^* \rangle\rangle$ is a language $\mathrm{supp}(r) = \{w \in \Sigma^* \mid (r, w) \neq 0\}$, and the *characteristic series* $\underline{L} \in S\langle\langle \Sigma^* \rangle\rangle$ of a language $L \subseteq \Sigma^*$ over a nontrivial semiring $S$ is given by $(\underline{L}, w) = 1$ for $w \in L$ and $(\underline{L}, w) = 0$ for $w \in \Sigma^* \setminus L$.

A *nondeterministic finite automaton* over an alphabet $\Sigma$ is a quadruple $\mathcal{A} = (Q, \to, I, F)$, where $Q$ is a finite set of states, $\to \subseteq Q \times \Sigma \times Q$ is a transition relation, and $I, F \subseteq Q$ are the sets of initial and final states, respectively. Given $p, q \in Q$ and $w \in \Sigma^*$ such that $w = a_1 \ldots a_n$ for some $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \Sigma$, we write $p \overset{w}{\to} q$ if there are states $p_0, \ldots, p_n \in Q$ such that $p_0 = p$, $p_n = q$, and $(p_{k-1}, a_k, p_k) \in \to$ for $k = 1, \ldots, n$. The *language* described by $\mathcal{A}$ is given by $\|\mathcal{A}\| = \{w \in \Sigma^* \mid \exists p \in I\ \exists q \in F : p \overset{w}{\to} q\}$. The automaton $\mathcal{A}$ is *deterministic* if $|I| = 1$ and $q = q'$ whenever $p \overset{a}{\to} q$ and $p \overset{a}{\to} q'$ for some $p, q, q' \in Q$ and $a \in \Sigma$.

A *deterministic finite automaton* with a complete transition function over $\Sigma$ can also be written as $\mathcal{A} = (Q, \cdot, i, F)$, where $\cdot \colon Q \times \Sigma^* \to Q$ is a right action of $\Sigma^*$ on $Q$, where $i \in Q$ is the initial state, and $F \subseteq Q$ is a set of final states. The *language* recognised by $\mathcal{A}$ is given by $\|\mathcal{A}\| = \{w \in \Sigma^* \mid i \cdot w \in F\}$.

A *weighted automaton* $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over a semiring $S$ and alphabet $\Sigma$ is given by a finite set of states $Q$, a transition weighting function $\sigma \colon Q \times \Sigma \times Q \to S$, and functions $\iota \colon Q \to S$, $\tau \colon Q \to S$ assigning initial and final weights to states. A *run* of $\mathcal{A}$ is a word $\gamma = q_0 a_1 q_1 a_2 q_2 \ldots q_{t-1} a_t q_t$ with $t \in \mathbb{N}$, $q_0, \ldots, q_t \in Q$, and $a_1, \ldots, a_t \in \Sigma$ such that $\sigma(q_{k-1}, a_k, q_k) \neq 0$ for $k = 1, \ldots, t$; we say that $\gamma$ is a run on $w = a_1 \ldots a_t$ leading from $q_0$ to $q_t$. Given any such run $\gamma$, we set $\sigma(\gamma) = \sigma(q_0, a_1, q_1) \ldots \sigma(q_{t-1}, a_t, q_t)$ and $\overline{\sigma}(\gamma) = \iota(q_0)\sigma(\gamma)\tau(q_t)$. Let $\mathcal{R}(\mathcal{A}, w)$ be the set of all runs of $\mathcal{A}$ on $w \in \Sigma^*$. The *series* $\|\mathcal{A}\| \in S\langle\langle \Sigma^* \rangle\rangle$ *realised by* $\mathcal{A}$ is then given by $(\|\mathcal{A}\|, w) = \sum_{\gamma \in \mathcal{R}(\mathcal{A}, w)} \overline{\sigma}(\gamma)$ for all $w \in \Sigma^*$. A series $r \in S\langle\langle \Sigma^* \rangle\rangle$ is *rational* over $S$ if $r = \|\mathcal{A}\|$ for some weighted automaton $\mathcal{A}$ over $S$.

Every $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over $S$ and $\Sigma$ such that $Q = [n]$ for some $n \in \mathbb{N}$ (which can always be assumed) determines a *linear representation* $\mathcal{P}_\mathcal{A} = (n, \mathbf{i}, \mu, \mathbf{f})$, where $\mathbf{i} = (\iota(1), \ldots, \iota(n))$ is a row vector of initial weights, $\mu$ is a homomorphism from $\Sigma^*$ to the monoid $S^{n \times n}$ of all $n \times n$ matrices over $S$ with matrix multiplication such that $\mu(a) = (\sigma(i, a, j))_{n \times n}$ for all $a \in \Sigma$, and $\mathbf{f} = (\tau(1), \ldots, \tau(n))^T$ is a column vector of final weights. One then has $(\|\mathcal{A}\|, w) = \mathbf{i}\mu(w)\mathbf{f}$ for all $w \in \Sigma^*$.

Let $\mathcal{A}_k = (Q_k, \sigma_k, \iota_k, \tau_k)$ be weighted automata over $S$ and $\Sigma$ for $k = 1, \ldots, n$, where $n \in \mathbb{N}$. The *disjoint union* of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is an automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$, where $Q = \bigcup_{k=1}^n (Q_k \times \{k\})$, $\sigma((p, k), a, (q, k)) = \sigma_k(p, a, q)$, $\iota(p, k) = \iota_k(p)$, and $\tau(q, k) = \tau_k(q)$ for $k = 1, \ldots, n$ and all $p, q \in Q_k$ and $a \in \Sigma$; moreover, $\sigma((p, k), a, (q, \ell)) = 0$ for all $k, \ell \in [n]$ such that $k \neq \ell$, and all $p \in Q_k$, $q \in Q_\ell$, and $a \in \Sigma$. Clearly $\|\mathcal{A}\| = \|\mathcal{A}_1\| + \ldots + \|\mathcal{A}_n\|$.

We also use some elementary concepts from *algebraic language theory*; we refer the reader to [44, 50] for the basic theory of *varieties of languages*, and to [1] for the theory of *pseudovarieties of finite monoids*.

## 3 Reversible Weighted Automata

We now define the *reversible weighted automata* over a semiring $S$ by generalising the notion of reversible finite automata as understood by J.-É. Pin [43].

**Definition 1.** *Let $S$ be a semiring and $\Sigma$ an alphabet. A weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over $S$ and $\Sigma$ is* reversible *if the following two conditions are satisfied for all $p, p', q, q' \in Q$ and $a \in \Sigma$:*

*(i) If $\sigma(p, a, q)$ and $\sigma(p, a, q')$ are both nonzero, then $q = q'$;*
*(ii) If $\sigma(p, a, q)$ and $\sigma(p', a, q)$ are both nonzero, then $p = p'$.*

The condition $(i)$ says that the transitions of $\mathcal{A}$ are deterministic; as there still can be multiple initial states, this is equivalent to saying that $\mathcal{A}$ is *finitely sequential* [8, 31, 33, 39–41]. The condition $(ii)$ says the same for the transpose of $\mathcal{A}$. If in addition there is at most one state $p \in Q$ with $\iota(p) \neq 0$ and at most one state $q \in Q$ with $\tau(q) \neq 0$, the automaton $\mathcal{A}$ is called *bideterministic* [29, 32].

We call a series over a semiring $S$ and over an alphabet $\Sigma$ *reversible* over $S$ if it is realised by some reversible weighted automaton over $S$ and $\Sigma$. The set of all reversible series over $S$ and $\Sigma$ is denoted by $\mathrm{RevS}(S, \Sigma)$. Moreover, by a *reversible language over $S$ and $\Sigma$*, we understand a support of some reversible series over $S$ and $\Sigma$; we write $\mathrm{RevL}(S, \Sigma) = \{\mathrm{supp}(r) \mid r \in \mathrm{RevS}(S, \Sigma)\}$ for the set of all such languages, and $\mathrm{RevL}(S)$ for the class of all reversible languages over $S$ and any alphabet. Observe that the reversible languages over the Boolean semiring $\mathbb{B}$ are precisely the usual reversible languages in the sense of J.-É. Pin [43]. We thus denote the positive variety of all such languages by $\mathrm{RevL}(\mathbb{B})$.

*Example 2.* We mostly consider a very special class of reversible weighted automata over *finite* – or slightly more generally, *locally finite* – *commutative rings* in this article. What can be readily observed right now is that given any nontrivial locally finite ring $R$, the class $\mathrm{RevL}(R)$ of all reversible languages over $R$ contains at least some languages that are not reversible in the usual sense, i.e., which do not belong to $\mathrm{RevL}(\mathbb{B})$. Indeed, let $R$ be of characteristic $n \geq 2$. Then the reversible weighted automaton $\mathcal{A}$ in Fig. 1 clearly realises the series

$$\|\mathcal{A}\| = \sum_{t \in \mathbb{N} \setminus \{0\}} a^t.$$

As a consequence, $\mathrm{supp}(\|\mathcal{A}\|) = a^+$ is a reversible language over $R$, so that $a^+ \in \mathrm{RevL}(R, \{a\})$. On the other hand, $a^+ \notin \mathrm{RevL}(\mathbb{B}, \{a\})$, as it does not satisfy the characterisation of reversible languages from [43]; more precisely, the ordered syntactic monoid of $a^+$ does not satisfy the pseudoinequality $x^\omega \leq 1$.

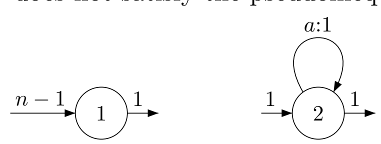


**Fig. 1:** A reversible weighted automaton $\mathcal{A}$ over a nontrivial locally finite ring $R$ of characteristic $n \geq 2$ and over a unary alphabet $\Sigma = \{a\}$.

Given the observation from the previous example, it seems to be worthwhile to take a closer look at the properties of the classes of reversible series and languages introduced above. At least a few observations can be made at the abstract level of semirings. Given any semiring $S$ and alphabet $\Sigma$, we denote by $1\mathrm{RevS}(S, \Sigma)$ the set of all series realised by reversible weighted automata $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over $S$ and $\Sigma$ with *precisely one initial state*, i.e., a state $q$ satisfying $\iota(q) \neq 0$.[1] We also write $1\mathrm{RevL}(S, \Sigma) = \{\mathrm{supp}(r) \mid r \in 1\mathrm{RevS}(S, \Sigma)\}$. We then easily obtain the following characterisation of reversible series over $S$.

**Proposition 3.** *Let $S$ be a semiring and $\Sigma$ an alphabet. Then* $\mathrm{RevS}(S, \Sigma)$ *consists of precisely all finite sums of series from* $1\mathrm{RevS}(S, \Sigma)$.

*Proof.* Let $\mathcal{A} = (Q, \sigma, \iota, \tau)$ be a reversible weighted automaton over $S$ and $\Sigma$. The automaton $\mathcal{A}_q = (Q, \sigma, \iota_q, \tau)$, with $\iota_q(q) = \iota(q)$ and $\iota_q(p) = 0$ for all $p \in Q \setminus \{q\}$, is then reversible with precisely one initial state for each $q \in Q$ such that $\iota(q) \neq 0$; as a result, $\|\mathcal{A}_q\| \in 1\mathrm{RevS}(S, \Sigma)$. Thus

$$\|\mathcal{A}\| = \sum_{\substack{q \in Q \\ \iota(q) \neq 0}} \|\mathcal{A}_q\|$$

is a finite sum of series from $1\mathrm{RevS}(S, \Sigma)$. Conversely, given $k \in \mathbb{N}$ and series $r_1, \ldots, r_k \in 1\mathrm{RevS}(S, \Sigma)$, each of the series $r_j$ for $j = 1, \ldots, k$ is realised by some reversible weighted automaton $\mathcal{A}_j$ over $S$ and $\Sigma$. The disjoint union of these automata is then clearly reversible as well, hence $r_1 + \ldots + r_k \in \mathrm{RevS}(S, \Sigma)$. □

[1] Note that such automata form a weighted generalisation of reversible deterministic finite automata in the sense of M. Holzer, S. Jakobi, and M. Kutrib [21, 22].

**Proposition 4.** *The set* $\mathrm{RevS}(S, \Sigma)$ *is closed under addition and under – both left and right – scalar multiplication for every semiring* $S$ *and alphabet* $\Sigma$.

*Proof.* Closure under addition follows directly by Proposition 3. For left or right scalar multiplication by $\alpha \in S$, it is clearly sufficient to multiply all initial or final weights of a reversible weighted automaton by $\alpha$. $\square$

Any finite automaton can be turned into a weighted automaton over a nontrivial semiring $S$ by assigning the weight 1 to all its transitions, initial states, and final states. It is standard that for *deterministic* finite automata recognising some language $L$, the resulting automaton realises the characteristic series $\underline{L}$ of $L$ over $S$. The following proposition records this observation specifically for the reversible automata with one initial state.

**Proposition 5.** *Let* $S$ *be a nontrivial semiring,* $L \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$*, and* $\underline{L}$ *be the characteristic series of* $L$ *over* $S$*. Then* $\underline{L} \in 1\mathrm{RevS}(S, \Sigma)$*. As a consequence,* $L \in 1\mathrm{RevL}(S, \Sigma)$ *and* $1\mathrm{RevL}(\mathbb{B}, \Sigma) \subseteq 1\mathrm{RevL}(S, \Sigma)$.

*Proof.* As $L \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$, the language $L$ is surely recognised by some reversible finite automaton $\mathcal{A} = (Q, \to, I, F)$ with precisely one initial state $i \in I$. Let $\mathcal{A}' = (Q, \sigma, \iota, \tau)$ be a weighted automaton such that for all $p, q \in Q$ and $a \in \Sigma$, one has $\sigma(p, a, q) = 1$ if $p \xrightarrow{a} q$ and $\sigma(p, a, q) = 0$ otherwise, $\iota(p) = 1$ if $p \in I$ and $\iota(p) = 0$ otherwise, and $\tau(q) = 1$ if $q \in F$ and $\tau(q) = 0$ otherwise. Since $\mathcal{A}$ is deterministic, surely $\|\mathcal{A}'\| = \underline{L}$, while the automaton $\mathcal{A}'$ is clearly reversible with precisely one initial state $i$. As a result, it follows that $\underline{L} \in 1\mathrm{RevS}(S, \Sigma)$ and $L = \mathrm{supp}(\underline{L}) \in 1\mathrm{RevL}(S, \Sigma)$. $\square$

## 4 The Case of $\mathbb{F}_2$

We now explore in detail the *reversible weighted automata* over the *two-element field* $\mathbb{F}_2$. The field $\mathbb{F}_2$ is arguably the most important finite ring from the viewpoint of weighted automata theory, and weighted automata over $\mathbb{F}_2$ can also be seen as natural devices for describing languages. Indeed, any formal power series over $\mathbb{F}_2$ is a characteristic series of its support, and both can be identified as a result. The rational series over $\mathbb{F}_2$ can thus be seen as rational languages, and the realisation of a rational series by a weighted automaton over $\mathbb{F}_2$ corresponds to recognising a language in "parity mode", where a word $w$ gets accepted when there is an odd number of successful runs on $w$ in the automaton. In fact, the theory of series over $\mathbb{F}_2$ viewed as languages has largely been developed under the name *formal languages over* GF(2) [7, 38], and weighted automata over $\mathbb{F}_2$ were also studied under the name *symmetric difference automata* [54, 55].

Similar observations can be made in particular for reversible weighted automata over $\mathbb{F}_2$: the series from $\mathrm{RevS}(\mathbb{F}_2, \Sigma)$ might be, for any alphabet $\Sigma$, identified with languages from $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$. It thus seems to be natural to ask about a characterisation of the language class $\mathrm{RevL}(\mathbb{F}_2)$ and about its relation to the usual class of reversible languages $\mathrm{RevL}(\mathbb{B})$. We prove in this section that $\mathrm{RevL}(\mathbb{F}_2)$ is in fact the *variety* generated by the positive variety $\mathrm{RevL}(\mathbb{B})$ – or equivalently, the *Boolean closure* of $\mathrm{RevL}(\mathbb{B})$.

Let us start by recording an easy observation that reversible weighted automata over $\mathbb{F}_2$ with *one initial state* realise precisely the characteristic series of languages recognised by reversible finite automata with one initial state.

**Proposition 6.** *Let $\Sigma$ be an alphabet and $\mathcal{A}$ a reversible weighted automaton over $\mathbb{F}_2$ and $\Sigma$ with one initial state. Then $\|\mathcal{A}\| = \underline{L}$ for some $L \in \mathrm{1RevL}(\mathbb{B}, \Sigma)$ over $\mathbb{F}_2$. As a consequence,* $\mathrm{1RevL}(\mathbb{F}_2, \Sigma) = \mathrm{1RevL}(\mathbb{B}, \Sigma)$.

*Proof.* Let $\mathcal{A} = (Q, \sigma, \iota, \tau)$ be a reversible weighted automaton with one initial state over $\mathbb{F}_2$ and $\Sigma$. As $\mathcal{A}$ is deterministic, the reversible finite automaton $\mathcal{A}' = (Q, \to, I, F)$ such that one has $p \xrightarrow{a} q$ for $p, q \in Q$ and $a \in \Sigma$ if and only if $\sigma(p, a, q) = 1$, while $I = \{p \in Q \mid \iota(p) = 1\}$ and $F = \{q \in Q \mid \tau(q) = 1\}$, clearly recognises $L = \mathrm{supp}(\|\mathcal{A}\|)$. Hence $L \in \mathrm{1RevL}(\mathbb{B}, \Sigma)$ and $\|\mathcal{A}\| = \underline{L}$. The equality $\mathrm{1RevL}(\mathbb{F}_2, \Sigma) = \mathrm{1RevL}(\mathbb{B}, \Sigma)$ then follows by Proposition 5. □

Recall from [6] that the class of languages recognised by reversible finite automata with precisely one initial state is closed under intersection. Indeed, if $\mathcal{A}_1 = (Q_1, \to_1, \{i_1\}, F_1)$ and $\mathcal{A}_2 = (Q_2, \to_2, \{i_2\}, F_2)$ with $i_1 \in Q_1$ and $i_2 \in Q_2$ are reversible finite automata over an alphabet $\Sigma$, then the finite automaton $\mathcal{A} = (Q_1 \times Q_2, \to, \{(i_1, i_2)\}, F_1 \times F_2)$, with $\to$ given for all $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$, and $a \in \Sigma$ by $(p_1, p_2) \xrightarrow{a} (q_1, q_2)$ if and only if $p_1 \xrightarrow{a}_1 q_1$ and $p_2 \xrightarrow{a}_2 q_2$, is clearly reversible as well. Let us record this observation for later reference.

**Proposition 7 (H. B. Axelsen, M. Holzer, and M. Kutrib [6]).** *The set* $\mathrm{1RevL}(\mathbb{B}, \Sigma)$ *is closed under intersection for every alphabet $\Sigma$.*

We can now explore some basic closure properties of the class $\mathrm{RevL}(\mathbb{F}_2)$ of reversible languages over $\mathbb{F}_2$. First of all, Proposition 4 directly implies closure of $\mathrm{RevL}(\mathbb{F}_2)$ under symmetric difference. Somewhat more interesting is the closure of this class under all Boolean operations, which we now establish.

**Proposition 8.** *Let $\Sigma$ be an alphabet. The set* $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$ *is then closed under the Boolean operations.*

*Proof.* Let $L \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$. Then $L$ is a support of some series in $\mathrm{RevS}(\mathbb{F}_2, \Sigma)$, which has to be the characteristic series $\underline{L}$ of $L$ over $\mathbb{F}_2$. Thus $\underline{L} \in \mathrm{RevS}(\mathbb{F}_2, \Sigma)$. Moreover, it is easy to see that the characteristic series $\underline{\Sigma^*}$ of $\Sigma^*$ over $\mathbb{F}_2$ is realised by a reversible weighted automaton over $\mathbb{F}_2$ and $\Sigma$ with one state, so that $\underline{\Sigma^*} \in \mathrm{RevS}(\mathbb{F}_2, \Sigma)$. It thus follows by Proposition 4 that the series

$$\underline{L} + \underline{\Sigma^*} = \sum_{w \in \Sigma^*} ((\underline{L}, w) + 1)\ w$$

belongs to $\mathrm{RevS}(\mathbb{F}_2, \Sigma)$. As a result,

$$\mathrm{supp}(\underline{L} + \underline{\Sigma^*}) = \{w \in \Sigma^* \mid (\underline{L}, w) + 1 \neq 0\} = \{w \in \Sigma^* \mid (\underline{L}, w) = 0\} = \Sigma^* \setminus L$$

belongs to $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$, and $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$ is closed under complementation.

Next, let $L, K \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$. Again, this means that the characteristic series $\underline{L}, \underline{K}$ over $\mathbb{F}_2$ are in $\mathrm{RevS}(\mathbb{F}_2, \Sigma)$. By Proposition 3 and Proposition 6, there are $m, n \in \mathbb{N}$ and $L_1, \ldots, L_m, K_1, \ldots, K_n \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$ such that over $\mathbb{F}_2$, one has $\underline{L} = \underline{L_1} + \ldots + \underline{L_m}$ and $\underline{K} = \underline{K_1} + \ldots + \underline{K_n}$. Now, for every $w \in \Sigma^*$,

$$\begin{aligned}(\underline{L \cap K}, w) &= (\underline{L_1} + \ldots + \underline{L_m}, w) \cdot (\underline{K_1} + \ldots + \underline{K_n}, w) \\ &= \big((\underline{L_1}, w) + \ldots + (\underline{L_m}, w)\big) \cdot \big((\underline{K_1}, w) + \ldots + (\underline{K_n}, w)\big) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (\underline{L_i}, w)(\underline{K_j}, w) = \sum_{i=1}^{m} \sum_{j=1}^{n} (\underline{L_i \cap K_j}, w)\end{aligned}$$

over $\mathbb{F}_2$, so that

$$\underline{L \cap K} = \sum_{i=1}^{m} \sum_{j=1}^{n} \underline{L_i \cap K_j}.$$

As $L_i \cap K_j \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$ by Proposition 7 for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, it follows by Proposition 5 that $\underline{L_i \cap K_j} \in 1\mathrm{RevS}(\mathbb{F}_2, \Sigma)$, and as a consequence, $\underline{L \cap K} \in \mathrm{RevS}(\mathbb{F}_2, \Sigma)$ by Proposition 3. As a result, $\mathrm{supp}(\underline{L \cap K}) = L \cap K$ has to be in $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$, which is thus closed under intersection.

Finally, the closure of $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$ under union follows by its closure under intersection and under complementation. □

As a first step towards a characterisation of the class $\mathrm{RevL}(\mathbb{F}_2)$ of reversible languages over $\mathbb{F}_2$, let us prove that this class contains all reversible languages in the usual sense – that is, all languages from $\mathrm{RevL}(\mathbb{B})$.

**Lemma 9.** *Let $\Sigma$ be an alphabet. Then $\mathrm{RevL}(\mathbb{B}, \Sigma) \subseteq \mathrm{RevL}(\mathbb{F}_2, \Sigma)$.*

*Proof.* Let $L \in \mathrm{RevL}(\mathbb{B}, \Sigma)$. Then, essentially by Proposition 3 applied in the case of the Boolean semiring, $L = L_1 \cup \ldots \cup L_n$ for some $n \in \mathbb{N}$ and languages $L_1, \ldots, L_n \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$. As $1\mathrm{RevL}(\mathbb{B}, \Sigma) = 1\mathrm{RevL}(\mathbb{F}_2, \Sigma)$ by Proposition 6 and as $1\mathrm{RevL}(\mathbb{F}_2, \Sigma) \subseteq \mathrm{RevL}(\mathbb{F}_2, \Sigma)$, we actually have $L_1, \ldots, L_n \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$. Thus $L = L_1 \cup \ldots \cup L_n$ has to be in $\mathrm{RevL}(\mathbb{F}_2, \Sigma)$ by Proposition 8. □

We are now finally ready to prove the characterisation of the class $\mathrm{RevL}(\mathbb{F}_2)$ as the Boolean closure of the positive variety of reversible languages $\mathrm{RevL}(\mathbb{B})$.

**Theorem 10.** *The class $\mathrm{RevL}(\mathbb{F}_2)$ is the Boolean closure of $\mathrm{RevL}(\mathbb{B})$.*

*Proof.* The Boolean closure $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$ of $\mathrm{RevL}(\mathbb{B})$ is included in $\mathrm{RevL}(\mathbb{F}_2)$ by Lemma 9 and Proposition 8. For the converse, let $L \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$ for some alphabet $\Sigma$. Proposition 3 gives us $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in 1\mathrm{RevS}(\mathbb{F}_2, \Sigma)$ such that $L = \mathrm{supp}(r_1 + \ldots + r_n)$. We prove that $L$ is in $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$ by induction on $n$. If $n = 0$, then $L = \mathrm{supp}(0) = \emptyset$ is in $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$. Now, for any $k \in \mathbb{N}$, $n = k+1$, and $L_1 := \mathrm{supp}(r_1 + \ldots + r_k)$ in $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$, we see that $L_2 := \mathrm{supp}(r_{k+1})$ is in $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$, as it in fact belongs to $1\mathrm{RevL}(\mathbb{B}, \Sigma)$ by Proposition 6. Thus

$$L = \mathrm{supp}((r_1 + \ldots + r_k) + r_{k+1}) = (L_1 \cup L_2) \cap (\Sigma^* \setminus (L_1 \cap L_2))$$

belongs to $\mathbf{B}(\mathrm{RevL}(\mathbb{B}))$ as well. □

The class $\mathrm{RevL}(\mathbb{F}_2)$ of all reversible languages over the field $\mathbb{F}_2$ thus forms a *variety of languages* given by the Boolean closure of the positive variety $\mathrm{RevL}(\mathbb{B})$ of the usual reversible languages in the sense of J.-É. Pin [43]. In other words, $\mathrm{RevL}(\mathbb{F}_2)$ is the *variety generated by* $\mathrm{RevL}(\mathbb{B})$. We thus see that introducing weights from $\mathbb{F}_2$ to reversible finite automata leads to describing a larger class of languages with better closure properties.

In fact, the Boolean closure of the positive variety of reversible languages $\mathrm{RevL}(\mathbb{B})$ is known to correspond, via Eilenberg's correspondence, to the pseudovariety $\mathbf{ECom} = \mathbf{J_1} * \mathbf{G}$ of all finite monoids with commuting idempotents [18, 19, 37, 4, 42]. We thus obtain the following corollary, which can be seen both as a characterisation of the reversible languages over $\mathbb{F}_2$, as well as a new automata-theoretic characterisation of the pseudovariety **ECom**.

**Corollary 11.** *Let $\Sigma$ be an alphabet a $L \subseteq \Sigma^*$ a language. Then $L$ belongs to* $\mathrm{RevL}(\mathbb{F}_2)$ *if and only if $ef = fe$ for any two idempotents $e, f$ in the syntactic monoid $M_L$ of $L$ over $\Sigma^*$. Thus $L \in \mathrm{RevL}(\mathbb{F}_2)$ if and only if $M_L \in$* **ECom***.*

The pseudovariety **ECom** can also be described using the pseudoidentity $x^\omega y^\omega = y^\omega x^\omega$, which directly captures the property of commuting idempotents. In any case, the membership of a finite monoid in **ECom** is clearly decidable, which implies decidability of the membership of a rational language in $\mathrm{RevL}(\mathbb{F}_2)$.

## 5 Automata over Locally Finite Commutative Rings

In what follows, we show that the results just obtained for reversible weighted automata over $\mathbb{F}_2$ can actually be generalised to weighted automata over any nontrivial locally finite commutative ring.

We prove $\mathrm{RevL}(R) = \mathrm{RevL}(\mathbb{F}_2)$ for any such ring $R$, and in order to establish one of the inclusions between these two classes, we show that any characteristic series of a language from $\mathrm{RevL}(\mathbb{F}_2)$ over $R$ can be realised by a reversible weighted automaton over $R$. In fact, local finiteness and commutativity of $R$ are not necessary here.

**Lemma 12.** *Let $R$ be a nontrivial ring and $\Sigma$ an alphabet. Then for every $L \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$, the characteristic series $\underline{L}$ of $L$ over $R$ is in $\mathrm{RevS}(R, \Sigma)$, and $L$ is in $\mathrm{RevL}(R, \Sigma)$.*

*Proof.* Let $L \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$; it suffices to show that $\underline{L} \in \mathrm{RevS}(R, \Sigma)$ over $R$. As $L \in \mathrm{RevL}(\mathbb{F}_2, \Sigma)$, it follows by Proposition 3 and by Proposition 6 that there exists some $n \in \mathbb{N}$ and languages $L_1, \ldots, L_n \in 1\mathrm{RevL}(\mathbb{B}, \Sigma)$ such that $L = \mathrm{supp}(\underline{L_1} + \ldots + \underline{L_n})$ over $\mathbb{F}_2$. This means that $w \in \Sigma^*$ is in $L$ if and only if the set $X_w = \{i \in [n] \mid w \in L_i\}$ contains an odd number of elements. Thus the coefficients of the characteristic series $\underline{L}$ of $L$ over $R$ at $w \in \Sigma^*$ are given by

$$(\underline{L}, w) = \begin{cases} 1 & \text{if } |X_w| \text{ is odd,} \\ 0 & \text{if } |X_w| \text{ is even.} \end{cases} \tag{1}$$

With $-2 := -(1{+}1)$ in $R$, let us now consider the series $r \in R\langle\langle \Sigma^* \rangle\rangle$ defined by

$$r = \sum_{\emptyset \subsetneq X \subseteq [n]} (-2)^{|X|-1} \underline{\bigcap_{i \in X} L_i}$$

(in particular, note that we have $-2 = 0$ in case $R$ is a ring of characteristic 2). Then $r \in \mathrm{RevS}(R, \Sigma)$ by virtue of Proposition 7, Proposition 5, Proposition 3, and Proposition 4. In order to prove that $\underline{L} \in \mathrm{RevS}(R, \Sigma)$, we now show that actually $r = \underline{L}$. Indeed, for $\emptyset \subsetneq X \subseteq [n]$ fixed and any $w \in \Sigma^*$, we have $\left(\underline{\bigcap_{i \in X} L_i}, w\right) = 1$ if $X \subseteq X_w$ and $\left(\underline{\bigcap_{i \in X} L_i}, w\right) = 0$ otherwise, while there are exactly $\binom{|X_w|}{k}$ sets $X \subseteq X_w$ of each size $k = 1, \ldots, |X_w|$. As a result,

$$(r, w) = \sum_{k=1}^{|X_w|} \binom{|X_w|}{k} (-2)^{k-1}$$

for each $w \in \Sigma^*$ over $R$. However, over $\mathbb{Q}$ it holds that

$$\sum_{k=1}^{|X_w|} \binom{|X_w|}{k} (-2)^{k-1} = -\frac{1}{2} \left( \sum_{k=0}^{|X_w|} \binom{|X_w|}{k} (-2)^k - 1 \right) = \frac{1 - (-1)^{|X_w|}}{2}$$

by the Binomial theorem, which means that over $\mathbb{Z}$ we obtain

$$\sum_{k=1}^{|X_w|} \binom{|X_w|}{k} (-2)^{k-1} = \begin{cases} 1 & \text{if } |X_w| \text{ is odd,} \\ 0 & \text{if } |X_w| \text{ is even.} \end{cases}$$

Now, by applying the unique ring homomorphism from $\mathbb{Z}$ to $R$ to both sides, we see that also over $R$ we have

$$(r, w) = \sum_{k=1}^{|X_w|} \binom{|X_w|}{k} (-2)^{k-1} = \begin{cases} 1 & \text{if } |X_w| \text{ is odd,} \\ 0 & \text{if } |X_w| \text{ is even.} \end{cases}$$

Thus $r = \underline{L}$ by (1). □

We now essentially establish the remaining inclusion in case the nontrivial ring $R$ is *locally finite* and *commutative*.

**Lemma 13.** *Let $R$ be a nontrivial locally finite commutative ring, $\Sigma$ an alphabet, $L \in \mathrm{RevL}(R, \Sigma)$ a language, and $M_L$ the syntactic monoid of $L$ over $\Sigma^*$. Then the idempotents of $M_L$ commute, i.e., $M_L \in$* **ECom**.

*Proof.* As **ECom** is a pseudovariety of finite monoids and $M_L$ divides the transition monoid of any deterministic finite automaton recognising $L$, it suffices to describe *some* deterministic finite automaton $\mathcal{D}$ over $\Sigma$ recognising $L$ such that the idempotents in the transition monoid of $\mathcal{D}$ commute.

As $L \in \mathrm{RevL}(R, \Sigma)$, there is a reversible weighted automaton $\mathcal{A}$ over $R$ and $\Sigma$ with state set $[n]$ for some $n \in \mathbb{N}$ such that $L = \mathrm{supp}(\|\mathcal{A}\|)$; let $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, \mu, \mathbf{f})$

be its linear representation. For $k = 1, \dots, n$, let $\mathbf{e}_k = (a_1, \dots, a_n) \in R^n$ be a vector with $a_k = 1$ and $a_j = 0$ for all $j \in [n] \setminus \{k\}$. As $\mathcal{A}$ always has to be over some finitely generated subring of $R$, we may assume that $R$ is actually finite.

We may thus take $\mathcal{D} = (R^n, \cdot, \mathbf{i}, F)$, where $F = \{\mathbf{v} \in R^n \mid \mathbf{vf} \neq 0\}$, and $\mathbf{v} \cdot a = \mathbf{v}\mu(a)$ for all $\mathbf{v} \in R^n$ and $a \in \Sigma$. The automaton $\mathcal{D}$ then clearly recognises the language $L = \mathrm{supp}(\|\mathcal{A}\|)$ and its transition monoid can be represented as a monoid of all matrices $\mu(w)$ for $w \in \Sigma^*$ with matrix multiplication over $R$. By reversibility of $\mathcal{A}$, each of the matrices $\mu(a)$ for $a \in \Sigma$ contains at most one nonzero element in each row and column. The same property thus holds for all matrices $\mu(w)$ for $w \in \Sigma^*$. This means that if $\mu(w) = (a_{i,j})_{n \times n}$ is an idempotent, then $\mu(w)$ is a diagonal matrix, as $a_{i,j} \neq 0$ for some $i \neq j$ would imply $\mathbf{e}_i\mu(w) = a_{i,j}\mathbf{e}_j$ and $\mathbf{e}_i\mu(w)^2 = a_{i,j}\mathbf{e}_j\mu(w) \neq a_{i,j}\mathbf{e}_j$, as otherwise $a_{j,j}$ would have to be nonzero and the $j$-th column of $\mu(w)$ would contain two nonzero elements. This would mean $\mu(w)^2 \neq \mu(w)$ and $\mu(w)$ would not be idempotent. Any idempotent $\mu(w)$ is thus indeed a diagonal matrix, while diagonal matrices over commutative rings commute under matrix multiplication. □

**Theorem 14.** *The class* $\mathrm{RevL}(R)$ *is the Boolean closure of* $\mathrm{RevL}(\mathbb{B})$ *for every nontrivial locally finite commutative ring $R$.*

*Proof.* Follows by Lemma 12, Lemma 13, and Theorem 10 coupled with the correspondence of the Boolean closure of $\mathrm{RevL}(\mathbb{B})$ to **ECom**. □

Let us finally establish decidability of the existence of a reversible equivalent for weighted automata over effective locally finite commutative rings.

**Proposition 15.** *Let $R$ be a nontrivial finite commutative ring, $\Sigma$ an alphabet, and $r \in R\langle\langle \Sigma^* \rangle\rangle$ a series rational over $R$. Then $r \in \mathrm{RevS}(R, \Sigma)$ if and only if $\mathrm{supp}(r + x \cdot \underline{\Sigma^*}) \in \mathrm{RevL}(R, \Sigma)$ for all $x \in R$.*

*Proof.* If $r \in \mathrm{RevS}(R, \Sigma)$, then $r + x \cdot \underline{\Sigma^*} \in \mathrm{RevS}(R, \Sigma)$ for all $x \in R$ as well, hence $\mathrm{supp}(r + x \cdot \underline{\Sigma^*}) \in \mathrm{RevL}(R, \Sigma)$. If on the other hand $\mathrm{supp}(r + x \cdot \underline{\Sigma^*}) \in \mathrm{RevL}(R, \Sigma)$ for all $x \in R$, we see that $r = \sum_{x \in R} x \cdot \underline{(\Sigma^* \setminus \mathrm{supp}(r - x \cdot \underline{\Sigma^*}))} \in \mathrm{RevS}(R, \Sigma)$ by Theorem 14, Lemma 12, and Proposition 4 (observe that we have just expressed $r$ as a specific recognisable step function [13, 14]). □

**Corollary 16.** *The problem of checking reversibility of a rational series over an effective locally finite commutative ring $R$ is decidable.*

*Proof.* If $R$ is nontrivial and $\mathcal{A}$ is a weighted automaton over $R$, then $r = \|\mathcal{A}\|$ is a series over some finitely generated subring $T$ of $R$, which is finite by local finiteness of $R$. Provided $R$ is effective, one can compute all elements $x \in T$ and the syntactic monoids of the languages $\mathrm{supp}(r + x \cdot \underline{\Sigma^*})$. In case all these monoids turn out to be in **ECom**, we obtain $r \in \mathrm{RevS}(T, \Sigma)$ by Theorem 14 and Proposition 15; hence also $r \in \mathrm{RevS}(R, \Sigma)$. If at least one of these monoids is not in **ECom**, then Proposition 15 gives us $r \notin \mathrm{RevS}(T', \Sigma)$ for all finite subrings $T'$ of $R$ containing $T$. As $R$ is locally finite, this implies $r \notin \mathrm{RevS}(R, \Sigma)$. □

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