On Deterministic Weighted Automata[☆]

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Abstract

Two families of input-deterministic weighted automata over semirings are considered: purely sequential automata, in which terminal weights of states are either zero or unity, and sequential automata, in which states can have arbitrary terminal weights. The class of semirings over which all weighted automata admit purely sequential equivalents is fully characterised. A similar characterisation is proved for sequential automata under an assumption that all elements of the underlying semiring have finitely many multiplicative left inverses, which is in particular true for all commutative semirings and all division semirings.

Keywords: Weighted Automaton, Determinism, Sequentiality, Pure Sequentiality

1. Introduction

Classical nondeterministic finite automata admit a well known extension, in which each transition is weighted (typically) by an element of some semiring [7, 22]. Such automata – usually known as *weighted automata* – realise formal power series instead of recognising languages, and they have been studied extensively both from the theoretical point of view and in connection with their practical applications. The reader might consult [7] for an overview of some of the most important research directions.

For applications such as natural language processing [16, 20], input-determinism of weighted automata often turns out to be crucial. This basically means that the automaton has precisely one state with nonzero initial weight and there is at most one transition for each input symbol leading from each state. However, it is well known that not all weighted automata can be determinised [19, 21]. The research has thus focused mainly on providing sufficient conditions – both on the automaton and the underlying semiring – under which a weighted automaton admits a deterministic equivalent, and on devising efficient determinisation algorithms for automata satisfying such conditions [1, 15, 20, 21].

We shall focus here on a slightly different question: over which semirings *all* weighted automata can be determinised? This in fact amounts to the study of deterministic weighted automata from a negative point of view, as we shall see that the class of such semirings is fairly constrained.

More precisely, we shall deal with this question for two classes of deterministic weighted automata: for *purely sequential* weighted automata, in which terminal weights of states might only be chosen as zero or unity of the underlying semiring, and for *sequential* weighted automata, in which terminal weights can be arbitrary (the term "deterministic weighted automata" usually refers to the latter [21]). This terminology follows Lombardy and Sakarovitch [19]; it may differ significantly in other sources (in particular, purely sequential automata are often called sequential, while sequential automata are called subsequential [20]).

We shall prove that weighted automata over S always admit *purely sequential* equivalents if and only if S is a locally finite division semiring. Moreover, local finiteness of S is known to be sufficient to guarantee that all weighted automata over S have *sequential* equivalents [19]. We shall prove that if S has no element with infinitely many multiplicative left inverses, then this is also a necessary condition. In particular, if S is commutative or a division semiring, then weighted automata over S can always be

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sequentialised if and only if S is locally finite.

Finally, let us mention that there is a branch of research motivated by quantitative formal verification dealing with weighted – or quantitative – automata over various structures beyond semirings [2, 3, 4, 5, 6, 8, 9, 10, 11, 18]. We shall nevertheless confine ourselves to the classical setting of semirings in this article, making theoretical analysis more tractable. Possible extensions of the results presented herein to structures more general than semirings are left for further research.

2. Preliminaries

A *monoid* is a triple $(M, \cdot, 1)$, where M is a set, $\cdot$ is an associative binary operation on M, and 1 is a neutral element with respect to $\cdot$. A *commutative monoid* is a monoid $(M, \cdot, 1)$ such that $\cdot$ is commutative. A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, the operation $\cdot$ distributes over $+$ both from left and from right, and $0 \cdot a = a \cdot 0 = 0$ holds for all a in S. We shall always assume that $S \neq \{0\}$. A *commutative semiring* is a semiring such that $(S, \cdot, 1)$ is commutative. A *division semiring* [12] is a (not necessarily commutative) semiring such that $(S - \{0\}, \cdot, 1)$ is a group, i.e., each nonzero element of S has a multiplicative inverse. We shall often simply write S for a semiring $(S, +, \cdot, 0, 1)$.

A *subsemiring* of a semiring $(S, +, \cdot, 0, 1)$ is a semiring $(S', +, \cdot, 0, 1)$ such that $S' \subseteq S$ (and $+, \cdot$ are restricted to $S' \times S'$). If moreover $X \subseteq S$ is a set, the *subsemiring of S generated by X* is the intersection of all subsemirings of S containing X. A semiring S is *finitely generated* if it is generated by some finite subset of S. A semiring S is *locally finite* if every finitely generated subsemiring of S is finite. Submonoids, finitely generated monoids, and locally finite monoids are defined similarly.

A *formal power series* over a semiring S and over an alphabet Σ is a mapping $r\colon \Sigma^* \to S$. It is customary to write (r, w) instead of $r(w)$ for the value of r on a word w in Σ^*; the formal power series r itself is then written as

$$r = \sum_{w \in \Sigma^*} (r, w)w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle \Sigma^* \rangle\rangle$.

Let r_1 and r_2 be in $S\langle\langle \Sigma^* \rangle\rangle$. The series $r_1 + r_2$ is then defined by $(r_1 + r_2, w) = (r_1, w) + (r_2, w)$ for all w in Σ^* and the series $r_1 \cdot r_2$ is defined by

$$(r_1 \cdot r_2, w) = \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} (r_1, u)(r_2, v)$$

for all w in Σ^*. The set $S\langle\langle \Sigma^* \rangle\rangle$ constitutes a semiring together with these two operations [7].

Let S be a semiring. A *proper weighted automaton* [7] over S is a sextuple $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$, where Q is a nonempty finite set of states, Σ is a (finite) alphabet, $T \subseteq Q \times \Sigma \times Q$ is a set of transitions, $\nu\colon T \to S$ is a transition weighting function, $\iota\colon Q \to S$ is an initial weighting function, and $\tau\colon Q \to S$ is a terminal weighting function.

Moreover, a *run* in $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ is a word $\gamma = p_1[p_1, c_1, q_1]p_2[p_2, c_2, q_2] \ldots p_n[p_n, c_n, q_n]p_{n+1}$ in $(QT)^*Q$ such that n is a nonnegative integer and $q_i = p_{i+1}$ for $i = 1, \ldots, n$. We shall denote by $S_{\mathcal{A}}(\gamma) := p_1$ the *source* of the run γ, and by $D_{\mathcal{A}}(\gamma) := p_{n+1}$ the *destination* of γ. The *label* of a run γ in $\mathcal{A}$ is given by $\lambda_{\mathcal{A}}(\gamma)$, where $\lambda_{\mathcal{A}}\colon (Q \cup T)^* \to \Sigma^*$ is a homomorphism such that $\lambda_{\mathcal{A}}(q) = \varepsilon$ for all q in Q and $\lambda_{\mathcal{A}}([p, c, q]) = c$ for all $[p, c, q]$ in T. The *weight* of a run γ in $\mathcal{A}$ is given by $W_{\mathcal{A}}(\gamma)$, where $W_{\mathcal{A}}\colon (Q \cup T)^* \to (S, \cdot, 1)$ is a monoid homomorphism such that $W_{\mathcal{A}}(q) = 1$ for all q in Q and $W_{\mathcal{A}}([p, c, q]) = \nu([p, c, q])$ for all $[p, c, q]$ in T. If $\mathcal{A}$ is clear from the context, we shall usually write $S(\gamma)$, $D(\gamma)$, $\lambda(\gamma)$, and $W(\gamma)$ instead of $S_{\mathcal{A}}(\gamma)$, $D_{\mathcal{A}}(\gamma)$, $\lambda_{\mathcal{A}}(\gamma)$, and $W_{\mathcal{A}}(\gamma)$, respectively.

Let us denote by $R(\mathcal{A})$ the set of all runs in $\mathcal{A}$ and by $R(\mathcal{A}, w)$, where w is in Σ^*, the set of all γ in $R(\mathcal{A})$ such that $\lambda(\gamma) = w$. The *behaviour* of $\mathcal{A}$ then is a power series $\|\mathcal{A}\|$ defined by

$$(\|\mathcal{A}\|, w) = \sum_{\gamma \in R(\mathcal{A}, w)} \iota(S(\gamma))W(\gamma)\tau(D(\gamma))$$

for all w in Σ^*. Note that this sum is always finite, and thus well defined.

We shall always assume that $\nu([p, c, q]) \neq 0$ for all $[p, c, q]$ in T – this is without loss of generality, as having a transition with zero weight is clearly equivalent to having no transition at all.

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be a proper weighted automaton over a semiring S. The automaton $\mathcal{A}$ is *sequential* if q_0 in Q exists such that $\iota(q_0) \neq 0$ and $\iota(q) = 0$ for all $q \neq q_0$ in Q, and if *at most one* q in Q with $[p, c, q]$ in T exists for each p in Q and c in Σ. Moreover, $\mathcal{A}$ is *purely sequential* if it is sequential and if $\tau(q)$ is in $\{0, 1\}$ for each q in Q. Finally, $\mathcal{A}$ is *unambiguous* if $R(\mathcal{A}, w)$ contains at most one run γ with $\iota(S(\gamma))W(\gamma)\tau(D(\gamma)) \neq 0$ for each w in Σ^*.

Remark 2.1. A purely sequential automaton *can* have a state with *initial* weight equal neither to 0, nor to 1. However, it is easy to see that when it comes to series realised by purely sequential automata, restricting initial weights to 0 and 1 would only affect possible weights of the empty word.

Remark 2.2. We shall confine ourselves to the study of *proper* weighted automata, i.e., automata without transitions labelled by the empty word. This clearly has no effect on our results. Moreover, *all weighted automata are understood to be proper in what follows.*

It is well known that each weighted automaton over a locally finite semiring S admits a sequential equivalent [19]. The underlying construction is based on an observation that each such automaton realises a series over some finite subsemiring S' of S. It is thus possible to incorporate weights into the usual subset construction, while "emitting" them as terminal weights. Let us restate this property as a proposition for later reference.

Proposition 2.3. *Let S be a locally finite semiring and $\mathcal{A}$ be a weighted automaton over S. Then there exists a sequential automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

3. Locally Finite Semirings

We shall now gather some simple facts about locally finite semirings. First, let us observe that local finiteness of a semiring is in fact equivalent to local finiteness of its multiplicative monoid.

Proposition 3.1. *Let $(S, +, \cdot, 0_S, 1_S)$ be a semiring. Then S is locally finite if and only if the monoid $(S, \cdot, 1_S)$ is locally finite.*

Proof. The "only if" part of the statement is trivial. For the converse, let us assume that the monoid $(S, \cdot, 1_S)$ is locally finite. We shall prove that the semiring $(S, +, \cdot, 0_S, 1_S)$ is locally finite as well.

As the monoid $(S, \cdot, 1_S)$ is locally finite, the element $2_S := 1_S + 1_S$ has to be of finite multiplicative order in S. This means there are nonnegative integers r, q such that $r < q$ and $2_S^r = 2_S^q$.

Now, let $X \subseteq S$ be a finite set, and let us denote by $[X]$ the subsemiring of S generated by X. By commutativity of $+$ and distributivity, each element x of $[X]$ can be written as a finite sum

$$x = \sum_{i=1}^{n} m_i \times x_i, \tag{1}$$

where n and $m_1, \ldots, m_n$ are nonnegative integers, $x_1, \ldots, x_n$ are *distinct* elements of the submonoid of $(S, \cdot, 1_S)$ generated by X, and $m_i \times x_i$ is a shorthand for $\sum_{j=1}^{m_i} x_i$. As $x_1, \ldots, x_n$ are distinct, it follows by local finiteness of $(S, \cdot, 1_S)$ that n is bounded from above by a constant. Moreover, as

$$2^r \times y = 2_S^r \cdot y = 2_S^q \cdot y = 2^q \times y$$

holds for each y in S, the numbers $m_1, \ldots, m_n$ can always be assumed to be smaller than some constant as well. The set of all elements x representable by (1) is thus finite. As this set equals $[X]$ and X is an arbitrary finite subset of S, this proves that S is locally finite. □

An element a of a semiring S is of *finite multiplicative order* if two distinct nonnegative integers n, m do exist so that $a^n = a^m$. Using similar ideas as in the proof of Proposition 3.1, it is possible to show that local finiteness is equivalent to finite multiplicative order of all elements for *commutative* semirings [17].

Proposition 3.2. *Let S be a commutative semiring. Then S is locally finite if and only if each element of S is of finite multiplicative order.*

4. Purely Sequential Weighted Automata

We shall now fully characterise the class of semirings S such that all weighted automata over S admit *purely sequential* equivalents by proving that this is the case if and only if S is a locally finite division semiring. We shall also observe that this property remains true after restricting the universe to unambiguous automata: purely sequential automata over S are strictly less powerful than unambiguous weighted automata over S whenever they are strictly less powerful than (general) weighted automata over S.

Let us first prove that S being a locally finite division semiring is a sufficient condition. We shall do this by introducing a suitable modification of the classical subset construction [13].

Lemma 4.1. *Let S be a locally finite division semiring and $\mathcal{A}$ be a weighted automaton over S. Then there exists a purely sequential automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$, and let

$$X := \nu(T) \cup \iota(Q) \cup \tau(Q)$$

be the set of all weights in $\mathcal{A}$. Let $[X]$ be a subsemiring of S generated by X. Local finiteness of S then implies that $[X]$ is finite and it is clear that $(\|\mathcal{A}\|, w)$ is in $[X]$ for all w in Σ^*. Moreover, S is a division semiring, implying that each nonzero x in S has a multiplicative inverse x^{-1}.

Let us now construct a purely sequential weighted automaton $\mathcal{A}' = (Q', \Sigma, T', \nu', \iota', \tau')$ over S equivalent to $\mathcal{A}$. Let us take $Q' = [X]^Q$ for the set of states, and let the set of transitions T' consist of all triples $[\varphi, c, \psi]$ such that φ, ψ are in $[X]^Q$, c is in Σ, and

$$\psi(q) = \sum_{[p,c,q]\in T} \varphi(p)\nu([p, c, q])$$

holds for all q in Q. Let the transition weighting function ν' be given for all $[\varphi, c, \psi]$ in T' by

$$\nu'([\varphi, c, \psi]) = G[\varphi]^{-1} \cdot G[\psi],$$

where $G[\kappa]$ is defined for all κ in $[X]^Q$ by

$$G[\kappa] = \begin{cases} \sum_{q\in Q} \kappa(q)\tau(q) & \text{if } \sum_{q\in Q} \kappa(q)\tau(q) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Next, let the initial weighting function ι' be given by $\iota'(\varphi) = G[\varphi]$ for the[1] mapping $\varphi \colon Q \to [X]$ satisfying $\varphi(q) = \iota(q)$ for all q in Q and by $\iota'(\psi) = 0$ for all other mappings $\psi \colon Q \to [X]$. Finally, the terminal weighting function will be given by

$$\tau'(\varphi) = \begin{cases} 1 & \text{if } \sum_{q\in Q} \varphi(q)\tau(q) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for all φ in $Q' = [X]^Q$.

We leave to the reader the straightforward proof of the fact that $\|\mathcal{A}'\| = \|\mathcal{A}\|$. □

Let us now turn to the converse implication of our characterisation. We shall first prove that local finiteness of S is necessary in order for all (all unambiguous, in fact) weighted automata to have a purely sequential equivalent.

Lemma 4.2. *Let S be a semiring that is not locally finite. Then there exists an unambiguous weighted automaton $\mathcal{A}$ over S, for which there is no purely sequential weighted automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

[1]Note that the equality $\varphi = \iota$ might not hold from a strictly formal viewpoint, as these two functions can have different codomains.

Proof. Assume that the semiring $(S, +, \cdot, 0, 1)$ is not locally finite. By Proposition 3.1, this means that the monoid $(S, \cdot, 1)$ is not locally finite, and hence it contains a finite subset $X = \{s_1, \ldots, s_n\}$ such that the element 0 is not in X and the submonoid $\langle X\rangle$ generated by X is infinite.

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be an (obviously unambiguous) weighted automaton over S given by $Q = \{1, 2, 3, 4\}$, $\Sigma = \{a_1, \ldots, a_n, b\}$,

$$T = \{[1, a_i, 2], [3, a_i, 4] \mid i \in \{1, \ldots, n\}\} \cup \{[2, b, 1], [4, b, 3]\},$$

$\nu([1, a_i, 2]) = s_i$ for $i = 1, \ldots, n$, $\nu([2, b, 1]) = 1$, $\nu([3, a_i, 4]) = 1$ for $i = 1, \ldots, n$, $\nu([4, b, 3]) = 1$, $\iota(1) = \iota(3) = 1$, $\iota(2) = \iota(4) = 0$, $\tau(1) = \tau(4) = 0$, and $\tau(2) = \tau(3) = 1$. The diagram of $\mathcal{A}$ is depicted in Figure 1 (nonzero initial and terminal weights are understood to be 1, and thus they are not made explicit in the diagram).

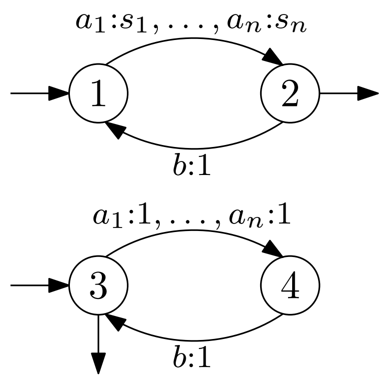

Figure 1: The diagram of the automaton $\mathcal{A}$.

It is easy to see that $(\|\mathcal{A}\|, a_{i_1} b a_{i_2} b \ldots b a_{i_k} b) = 1$ and $(\|\mathcal{A}\|, a_{i_1} b a_{i_2} b \ldots b a_{i_k} b a_{i_{k+1}}) = s_{i_1} \ldots s_{i_{k+1}}$ for every nonnegative integer k and every $i_1, \ldots, i_{k+1}$ in $\{1, \ldots, n\}$.

We shall prove that there is no purely sequential automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$. For contradiction, suppose that such automaton exists and let $\mathcal{A}' = (Q', \Sigma', T', \nu', \iota', \tau')$; obviously $\Sigma' \supseteq \Sigma$. Let x be in $\langle X\rangle - \{0, 1\}$. Then there is a nonnegative integer k and indices $i_1, \ldots, i_{k+1}$ in $\{1, \ldots, n\}$ such that $s_{i_1} \ldots s_{i_{k+1}} = x$. Let $w := a_{i_1} b a_{i_2} b \ldots b a_{i_k} b$ and $w' := w a_{i_{k+1}}$. By pure sequentiality of $\mathcal{A}'$, runs γ and γ' have to exist so that $R(\mathcal{A}', w) = \{\gamma\}$, $R(\mathcal{A}', w') = \{\gamma'\}$, and $\gamma' = \gamma[p, c, q]q$ for some $[p, c, q]$ in T'; in particular, $S(\gamma) = S(\gamma')$. Moreover, as $\mathcal{A}'$ is purely sequential, it follows that

$$\iota'(S(\gamma))W(\gamma) = (\|\mathcal{A}'\|, w) = 1$$

and

$$\iota'(S(\gamma))W(\gamma)\nu'([p,c,q]) = (\|\mathcal{A}'\|, w') = \\ = s_{i_1} \dots s_{i_{k+1}} = x.$$

Hence, $\nu'([p,c,q]) = x$. As x was chosen as an arbitrary element of an infinite set $\langle X \rangle - \{0,1\}$, this implies that $\mathcal{A}'$ has to have infinitely many transitions, i.e., a contradiction. ☐

We shall now prove that if all (all unambiguous) weighted automata over S admit purely sequential equivalents, then S has to be a division semiring.

Lemma 4.3. *Let S be a semiring that is not a division semiring. Then there exists an unambiguous weighted automaton $\mathcal{A}$ over S, for which there is no purely sequential weighted automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

Proof. Suppose that S is not a division semiring, i.e., $(S - \{0\}, \cdot, 1)$ is not a group. This means that the monoid $(S, \cdot, 1)$ contains a nonzero element s, which has no right inverse [14], i.e., s has no right multiplicative inverse in the semiring S.

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be an unambiguous automaton over S with $Q = \{1,2,3,4,5\}$, $\Sigma = \{a\}$, $T = \{[1,a,2],[3,a,4],[4,a,5]\}$, $\nu([1,a,2]) = s$, $\nu([3,a,4]) = \nu([4,a,5]) = 1$, $\iota(1) = \iota(3) = 1$, $\iota(2) = \iota(4) = \iota(5) = 0$, $\tau(1) = \tau(3) = \tau(4) = 0$, and $\tau(2) = \tau(5) = 1$. The diagram of the automaton $\mathcal{A}$ is depicted in Figure 2.

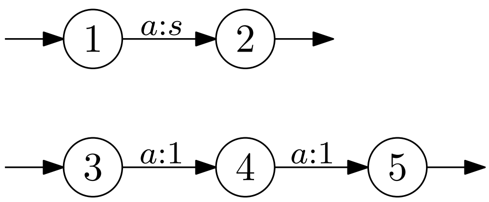

Figure 2: The diagram of the automaton $\mathcal{A}$.

It is clear that $(\|\mathcal{A}\|, a) = s$ and $(\|\mathcal{A}\|, aa) = 1$. It thus suffices to prove that there is no purely sequential automaton $\mathcal{A}'$ over S such that $(\|\mathcal{A}'\|, a) = s$ and $(\|\mathcal{A}'\|, aa) = 1$.

Suppose that $\mathcal{A}' = (Q', \Sigma', T', \nu', \iota', \tau')$ has this property; clearly, a has to be in Σ'. It then follows by pure sequentiality of $\mathcal{A}'$ that there is a run $\gamma = p_1[p_1,a,p_2]p_2[p_2,a,p_3]p_3$ in $\mathcal{A}'$ such that $\iota'(p_1)\nu'([p_1,a,p_2]) = s$ and $\iota'(p_1)\nu'([p_1,a,p_2])\nu'([p_2,a,p_3]) = 1$. Thus, $\nu'([p_2,a,p_3])$ is a right multiplicative inverse of s: a contradiction. ☐

The results obtained so far yield the following characterisation.

Theorem 4.4. *Let S be a semiring. The following statements are equivalent:*

(i) All weighted automata over S have purely sequential equivalents over S.

(ii) All unambiguous weighted automata over S have purely sequential equivalents over S.

(iii) S is a locally finite division semiring.

Proof. Follows directly by Lemma 4.1, Lemma 4.2, and Lemma 4.3. ☐

Remark 4.5. Although an arbitrarily large finite alphabet is used in the counterexample presented in the proof of Lemma 4.2, it is clear that a two-letter alphabet suffices if the original symbols are encoded and the reasoning is modified accordingly.

5. Sequential Weighted Automata

Let us now consider *sequential* weighted automata. We shall prove that if S is a semiring that contains no element with infinitely many multiplicative left inverses (in particular, this is the case when S is a commutative semiring or a division semiring), then all weighted automata over S have sequential equivalents if and only if S is locally finite. We shall leave open the question if the assumption of S not containing elements with infinitely many multiplicative left inverses can be weakened or abandoned.

Lemma 5.1. *Let S be a semiring that is not locally finite and contains no element with infinitely many multiplicative left inverses. Then there exists an unambiguous weighted automaton $\mathcal{A}$ over S, for which there is no sequential weighted automaton $\mathcal{A}'$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$.*

Proof. As the semiring $(S, +, \cdot, 0, 1)$ is not locally finite, it follows by Proposition 3.1 that the monoid $(S, \cdot, 1)$ is not locally finite. Hence, there is a finite subset $X = \{s_1, \dots, s_n\}$ of $S - \{0\}$ such that the submonoid $\langle X \rangle$ generated by X is infinite.

Let $\mathcal{A} = (Q, \Sigma, T, \nu, \iota, \tau)$ be the unambiguous weighted automaton from the proof of Lemma 4.2 (for convenience, the diagram of $\mathcal{A}$ is depicted once again in Figure 3).

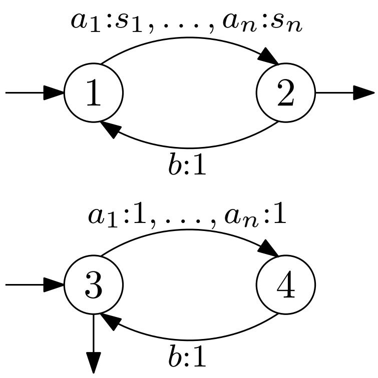

Figure 3: The diagram of the automaton $\mathcal{A}$.

Recall that $(\|\mathcal{A}\|, a_{i_1}ba_{i_2}b \dots ba_{i_k}b) = 1$ and $(\|\mathcal{A}\|, a_{i_1}ba_{i_2}b \dots ba_{i_k}ba_{i_{k+1}}) = s_{i_1} \dots s_{i_{k+1}}$ for every nonnegative integer k and every $i_1, \dots, i_{k+1}$ in $\{1, \dots, n\}$. Suppose for the purpose of contradiction that there is a sequential weighted automaton $\mathcal{A}' = (Q', \Sigma', T', \nu', \iota', \tau')$ over S such that $\|\mathcal{A}'\| = \|\mathcal{A}\|$; then clearly $\Sigma' \supseteq \Sigma$.

Moreover, let

$$L = \{a_{i_1}ba_{i_2}b \dots ba_{i_{k+1}} \mid i_1, \dots, i_{k+1} \in \{1, \dots, n\}\}$$

and let us denote

$$\overline{R}(\mathcal{A}') = \bigcup_{w \in L} R(\mathcal{A}', w).$$

As the monoid $\langle X \rangle$ is infinite and as each element of $\langle X \rangle - \{1\}$ is a coefficient in $\|\mathcal{A}'\|$ of some w in L, it follows that there has to be a state q in Q' such that the set

$$V(q) = \{\iota'(S(\gamma))W(\gamma) \mid \gamma \in \overline{R}(\mathcal{A}');\ D(\gamma) = q\}$$

is infinite. However, as $\mathcal{A}'$ is sequential, it is clear that each γ in $\overline{R}(\mathcal{A}')$ with $D(\gamma) = q$ can be extended to a run $\gamma' := \gamma[q, b, q']q'$, where q' is in Q' and

$$\iota'(S(\gamma))W(\gamma)\nu'([q, b, q'])\tau'(q') = 1.$$

As a result, each element of $V(q)$ is a multiplicative left inverse for $\nu'([q, b, q'])\tau'(q')$. As $V(q)$ is infinite, this contradicts our assumption on S. □

The lemma that we have just proved yields, with Proposition 2.3, the following characterisation.

Theorem 5.2. *Let S be a semiring that contains no element with infinitely many multiplicative left inverses. The following statements are equivalent:*

(*i*) *All weighted automata over S have sequential equivalents over S.*

(*ii*) *All unambiguous weighted automata over S have sequential equivalents over S.*

(*iii*) *S is locally finite.*

Proof. Follows directly by Proposition 2.3 and Lemma 5.1. □

In particular, let us mention explicitly that the theorem stated above gives us a complete characterisation of all *commutative semirings* and all *division semirings* S such that all weighted automata over S have sequential equivalents.

Corollary 5.3. *Let S be a commutative semiring or a division semiring. The following statements are equivalent:*

(*i*) *All weighted automata over S have sequential equivalents over S.*

(*ii*) *All unambiguous weighted automata over S have sequential equivalents over S.*

(*iii*) *S is locally finite.*

Proof. If S is commutative and $ba = ca = 1$, then $b = bca = bac = c$. In other words, each a in S has at most one multiplicative left inverse. If S is a division semiring, this property follows by the fact that $(S - \{0\}, \cdot, 1)$ is a group and 0 has no multiplicative left inverse. Hence, the claim follows by Theorem 5.2. □

Remark 5.4. Similarly as in the case of purely sequential automata, the counterexample used in Lemma 5.1 can obviously be modified so that a two-letter alphabet is used instead of an arbitrarily large finite alphabet.

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