Rational Elements of Summation Semirings*

Peter Kostolányi

Department of Computer Science, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

Abstract

The theory of finite automata and rational semiring elements is reconsidered in the setting of summation semirings, traditionally known as Σ-semirings. These generalise complete semirings by allowing infinite sums to be defined just for selected families of elements. A relation of the presented approach to the theory of finite automata over partial Conway semirings is discussed. Equivalence of finite automata over summation semirings to rational expressions and right-linear systems is proved under suitable semantics. Moreover, MSO logics over summation semirings are introduced and proved to be equivalent to finite automata.

Keywords: Rational Element, Summation Semiring, Finite Automaton, MSO Logic

1. Introduction

Summation semirings form a generalisation of complete semirings with infinite sums defined just for some families of elements called *summable families*. These semirings have been called Σ*-semirings* by Hebisch and Weinert [14]. They have been renamed to summation semirings in [17] to avoid notational confusion when Σ also denotes an alphabet.

Each (countably) complete semiring is a summation semiring, in which all finite or (countably) infinite families of elements are summable. Moreover, each semiring of formal power series can be turned into a summation semiring with all locally finite families of series being summable. Summation semirings thus appear to be a natural unifying framework for a large portion of automata theory over semirings, as restriction to complete semirings (or their specialisations) and restriction to locally finite families of series (usually via some notion of "properness" for automata or systems of equations) are probably the most commonplace approaches to handling infinite sums in this area [7].

Summation semirings have already been considered as a basis for a unified theory of *algebraic* semiring elements [17]. The choice between constraining the universe of semirings and constraining the universe of algebraic systems or pushdown automata, which traditionally needed to be made when developing a theory of algebraic semiring elements or algebraic power series [18, 21, 24], is no longer a necessary nuisance in this unified setting. Moreover, the semantics of algebraic systems and pushdown automata over summation semirings has been defined so that certain properties of context-free languages can simply be "lifted" to algebraic summation semiring elements [17].

The scope of a summation-semiring-based approach seems not to be limited to algebraic phenomena – the author believes that summation semirings can potentially serve as a unifying basis for a large portion of automata theory over semirings. The aim of this article is to partially support this claim by developing a theory of *finite automata* over semirings and *rational* semiring elements using summation semirings as the underlying framework. Similarly as in the case of algebraic systems [17], the theory developed in this article will unify the traditional approaches based on finite automata over complete semirings and on proper finite automata over arbitrary semirings of formal power series.

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Email address: kostolanyi@fmph.uniba.sk (Peter Kostolányi)

It has to be noted that a unification of these two traditional approaches is already provided by the theory of finite automata over partial Conway semirings over an ideal, introduced by Bloom, Ésik, and Kuich [2, 12]. We shall relate both unifying theories by showing that their scopes are incomparable, though it seems that all settings really interesting from the automata-theoretic perspective can be captured both via summation semirings and via partial Conway semirings. The author nevertheless believes that summation semirings can serve as a useful complement to partial Conway semirings, mainly for the following two reasons:

1. The use of summation semirings is not limited to the study of rational elements. Algebraic semiring elements have already been studied from this perspective [17] and it seems that other parts of automata theory can be built upon summation semirings as well. It is thus for instance possible to study finite automata and some more powerful models consistently in the same framework, while still incorporating all complete semirings and all "proper" automata over arbitrary power series semirings.
2. Transitions from formal languages to more general semirings via homomorphisms become a natural proof technique when summation semirings are used as an underlying framework. As we shall see, this makes it possible to prove expressive equivalence of several different models describing rational summation semiring elements by utilising the fact that the equivalence holds over the semiring of formal languages. We shall also obtain a definition of MSO logics over semirings using this approach.

Apart from automata, we shall also define rational expressions and right-linear systems over summation semirings and prove equivalence of all these models under suitable semantics. Nevertheless, some of the usual properties of rational elements do not generalise to the setting of summation semirings. This is mainly due to the fact that certain infinite sums, which are typically defined in the classical settings, can be undefined over summation semirings.

Moreover, we shall introduce monadic second-order (MSO) logics over summation semirings such that MSO-definability coincides with rationality. Note that these logics will *not* generalise weighted MSO logics of Droste and Gastin [4, 5]. While weighted MSO logics extend classical MSO logics on words by the possibility of "emitting" weights, MSO logics over semirings are no longer logics on words, but "logics on factorisations of semiring elements". For this reason, the specialisation of MSO logics over summation semirings to semirings of formal power series results in a new notion of MSO-definable series that should be distinguished from MSO-definability of series via weighted MSO logics. In particular, MSO-definable series as understood in this article are always rational, while this property does not hold without further restrictions for series definable by weighted MSO logics [4, 5].

2. Preliminaries

If not stated otherwise, an *alphabet* is understood to be a *finite* nonempty set. We shall occasionally work with infinite alphabets as well. The symbol $\mathbb{N}$ denotes the set of natural numbers *including* zero.

A (commutative) *semigroup* is a pair $(X, \cdot)$, where X is a set and $\cdot$ is a (commutative and) associative binary operation on X. A (commutative) *monoid* is a triple $(M, \cdot, 1)$, where $(M, \cdot)$ is a (commutative) semigroup and 1 is a neutral element with respect to $\cdot$, i.e., $1 \cdot a = a \cdot 1 = a$ holds for all a in M. A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$, where $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, the distributive laws $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$ hold for all a, b, c in S, and $0 \cdot a = a \cdot 0 = 0$ holds for all a in S. We shall often write S instead of $(S, +, \cdot, 0, 1)$. A *subsemiring* of a semiring S is a set $T \subseteq S$ containing 0 and 1 that is closed under both semiring operations. Without loss of generality, we shall assume that no semiring contains $\perp$ as an element; we shall later use this symbol to denote undefined values.

Let S be a set of elements and I a set of indices. A *family* $(a_i \mid i \in I)$ of elements of S indexed by I is a mapping $\varphi \colon I \to S$ such that $\varphi(i) = a_i$ for all i in I. Moreover, we shall say that a family $(a_i \mid i \in I)$ is of cardinality κ if I is of cardinality κ; a finite family is then a family of finite cardinality, a countably infinite family is a family of cardinality *precisely* $\aleph_0$, an infinite family is a family of cardinality *at least* $\aleph_0$, etc. We shall write $\mathcal{F}(S)$ for the class[1] of all families of elements of S.

[1]Note that $\mathcal{F}(S)$ is not a set, as a family can be indexed by an arbitrary set.

A *generalised partition* of a set I is a family $(I_j \mid j \in J)$ of subsets of I such that $I = \bigcup_{j\in J} I_j$ and $I_j \cap I_k = \emptyset$ holds for all j, k in J such that $j \neq k$. A *partition* of a set I is a generalised partition $(I_j \mid j \in J)$ of I such that $I_j \neq \emptyset$ for all j in J.

We are now prepared to give a definition of *summation semirings*. These are called Σ-*semirings* in [14]; similarly as in [17], we shall use an alternative term "summation semirings" in order to avoid the clash of notation in case Σ also denotes an alphabet.

Definition 2.1. A *summation semiring* is a triple $(S, \mathcal{F}, \Phi)$ with S being a semiring, $\mathcal{F}$ a nonempty subclass of $\mathcal{F}(S)$ consisting of *summable families*, and $\Phi \colon \mathcal{F} \to S$ a mapping

$$\Phi \colon (a_i \mid i \in I) \mapsto \sum_{i \in I} a_i$$

such that the following conditions are satisfied:

(*i*) Let n be a nonnegative integer and $I = \{i_1, \ldots, i_n\}$ a finite set with n elements. Then each family $(a_i \mid i \in I)$ in $\mathcal{F}(S)$ indexed by I is in $\mathcal{F}$ and

$$\sum_{i \in I} a_i = a_{i_1} + \ldots + a_{i_n}.$$

(*ii*) Let $(a_i \mid i \in I)$ be in $\mathcal{F}$ and $(I_j \mid j \in J)$ a generalised partition of the index set I such that $|J| \leq \kappa$ for some κ that is a cardinality of at least one family in $\mathcal{F}$. Then the families $(a_i \mid i \in I_j)$ for j in J and the family $\left(\sum_{i\in I_j} a_i \;\middle|\; j \in J\right)$ are all in $\mathcal{F}$ and

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

(*iii*) Let $(a_i \mid i \in I)$ in $\mathcal{F}(S)$ be a family, J a finite set, and $(I_j \mid j \in J)$ a generalised partition of I such that the families $(a_i \mid i \in I_j)$ for j in J are all in $\mathcal{F}$. Then $(a_i \mid i \in I)$ is in $\mathcal{F}$ and thus, by (*i*) and (*ii*),

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

(*iv*) Let $(a_i \mid i \in I)$ and $(b_j \mid j \in J)$ be in $\mathcal{F}$. Then the family $(a_i \cdot b_j \mid (i,j) \in I \times J)$ is in $\mathcal{F}$ as well and

$$\left(\sum_{i \in I} a_i \right) \cdot \left(\sum_{j \in J} b_j \right) = \sum_{(i,j) \in I \times J} (a_i \cdot b_j).$$

By (*i*), sums over finite families always exist and they are consistent with addition in the semiring S. The condition (*ii*) essentially states that infinite sums are "associative" and "commutative". Finite sums of infinite sums can be expressed as a single infinite sum by (*iii*), and (*iv*) is a form of "infinite distributivity".

Example 2.2. Each *complete semiring* [6, 14] is a summation semiring $(S, \mathcal{F}, \Phi)$ such that $\mathcal{F} = \mathcal{F}(S)$. Similarly, each *countably complete semiring* [14] is a summation semiring $(S, \mathcal{F}, \Phi)$ such that $\mathcal{F}$ consists of all finite and countably infinite families in $\mathcal{F}(S)$. Proofs can be found in [14, 17].

Example 2.3. Let S be a semiring and Σ an alphabet. A (noncommutative) *formal power series* over S and Σ [1, 6, 24] is a mapping $r \colon \Sigma^* \to S$. The value $r(w)$ is then usually denoted by (r, w) and called the *coefficient* of w in r, while the series r is written as

$$r = \sum_{w \in \Sigma^*} (r, w) w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle\Sigma^*\rangle\rangle$. If r_1 and r_2 are series in $S\langle\langle\Sigma^*\rangle\rangle$, then one can define the series $r_1 + r_2$ (the *sum* of r_1 and r_2) for each w in Σ^* by

$$(r_1 + r_2, w) = (r_1, w) + (r_2, w)$$

and the series $r_1 \cdot r_2$ (the *Cauchy product* of r_1 and r_2) for each w in Σ^* by

$$(r_1 \cdot r_2, w) = \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} (r_1, u) \cdot (r_2, v).$$

Moreover, let us denote by 0 the series such that $(0, w) = 0$ for each w in Σ^* and by 1 the series such that $(1, \varepsilon) = 1$ and $(1, w) = 0$ for each w in Σ^+. The quintuple $(S\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, 1)$ is then a semiring [1, 6, 24] called the *semiring of formal power series* over S and Σ, usually denoted simply by $S\langle\langle\Sigma^*\rangle\rangle$.

A family $(r_i \mid i \in I)$ in $\mathcal{F}(S\langle\langle\Sigma^*\rangle\rangle)$ is called *locally finite* if the set $I(w) = \{i \in I \mid (r_i, w) \neq 0\}$ is finite for each w in Σ^*. The semiring $S\langle\langle\Sigma^*\rangle\rangle$ can then be turned to a summation semiring $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$ such that $\mathcal{F}$ consists of all locally finite families in $\mathcal{F}(S)$ and

$$\Phi(r_i \mid i \in I) = \sum_{i \in I} r_i$$

is defined for each locally finite family $(r_i \mid i \in I)$ in $\mathcal{F}$ by

$$\sum_{i \in I} r_i = r,$$

where r is the series such that

$$(r, w) = \sum_{i \in I(w)} (r_i, w)$$

for each w in Σ^*. A proof that this indeed is a summation semiring can be found in [17].

3. Finite Automata over Summation Semirings

We shall now define *finite automata over summation semirings* as a generalisation of both finite automata over complete semirings [11] and proper automata [22, 23] over arbitrary power series semirings (i.e., proper *weighted automata* [7] over an arbitrary semiring of coefficients).[2]

Similarly as for algebraic systems or pushdown automata over summation semirings [17], the definition of finite automata themselves poses no actual problem – the fact that the underlying semiring is a summation semiring is not even used there. This property becomes important only in the definition of the *behaviour* of a finite automaton, which is the core of the following considerations. The behaviour of a finite automaton $\mathcal{A}$ over a summation semiring S will be defined via an auxiliary automaton over the semiring of formal languages over a suitably chosen alphabet, which recognises "transcripts" of all valid runs of the original automaton $\mathcal{A}$. We shall call this auxiliary automaton the *template automaton* of $\mathcal{A}$ and define the behaviour of $\mathcal{A}$ to be a semiring element obtained from the language recognised by its template automaton by applying a suitable homomorphism to its words and summing the homomorphic images up. Note that a similar homomorphism is also used in the generalised Nivat's theorem for weighted automata [8].

Definition 3.1. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. A *finite S'-automaton* over S is a quadruple $\mathcal{A} = (Q, \iota, T, \tau)$, where Q is a nonempty finite set of states, $\iota\colon Q \to S'$ is an initial weighting function, $T \subseteq Q \times S' \times Q$ is a *finite* set of transitions in $\mathcal{A}$, and $\tau\colon Q \to S'$ is a terminal weighting function.

[2] The definition below will *not* generalise cycle-free weighted automata of [11]. However, rather than by inherent limitations of summation semirings, this is caused by a peculiar reason that we shall return to in Section 8. We shall see there that it is possible to incorporate cycle-free automata for a slightly different definition of automata semantics.

In what follows, we shall denote by $2_1^X = \binom{X}{1}$ the set of all singleton subsets of a set X and identify each $\{x\}$ in 2_1^X directly with x in X. Moreover, the following definition makes use of the notion of a language $\|\mathcal{B}\|$ recognised by a finite $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-automaton $\mathcal{B} = (Q, \iota, T, \tau)$ over 2^{Σ^*} for some alphabet Σ. This is defined in the usual way except that an element of $2_1^{\Sigma^*}$ (identified with a word from Σ^*) "read" by $\mathcal{B}$ during a run γ leading from state p to q should be prefixed by $\iota(p)$ and suffixed by $\tau(q)$. More precisely, a *run* of $\mathcal{B}$ is a word $\gamma = q_0 c_1 q_1 \dots c_n q_n$ with n in $\mathbb{N}$, $q_0, \dots, q_n$ in Q, and $c_1, \dots, c_n$ in $2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\}$, such that (q_{i-1}, c_i, q_i) is in T for $i = 1, \dots, n$. The *label* of γ is the element of 2^{Σ^*} defined by $\|\gamma\| := c_1 \dots c_n$; it is clear that $\|\gamma\|$ always belongs to $2_1^{\Sigma^*} \cup \{\emptyset\}$. Moreover, let us write $\sigma(\gamma) := q_0$ for the *source* of the run $\gamma = q_0 c_1 q_1 \dots c_n q_n$, and $\nu(\gamma) := q_n$ for the *destination* of γ. Let us denote by $R(\mathcal{B})$ the set of all runs of the automaton $\mathcal{B}$. The language $\|\mathcal{B}\|$ recognised by $\mathcal{B}$ is then defined by

$$\|\mathcal{B}\| := \bigcup_{\gamma \in R(\mathcal{B})} \iota(\sigma(\gamma)) \|\gamma\| \tau(\nu(\gamma)).$$

Definition 3.2. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and $\mathcal{A} = (Q, \iota, T, \tau)$ a finite S'-automaton over S. Let us denote by Σ the alphabet $\Sigma = (Q \times \{1, 2\}) \cup T$. A *template automaton* of $\mathcal{A}$ is then a finite $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-automaton $\mathrm{temp}(\mathcal{A}) = (Q, \iota', T', \tau')$ over 2^{Σ^*} defined by $\iota'(q) = (q, 1)$ for each q in Q,

$$T' = \{(p, (p, a, q), q) \mid (p, a, q) \in T\},$$

and $\tau'(q) = (q, 2)$ for each q in Q. Let $\|\mathrm{temp}(\mathcal{A})\|$ be the language recognised by the automaton $\mathrm{temp}(\mathcal{A})$. Let $h[\mathcal{A}] \colon \Sigma^* \to (S, \cdot)$ be a monoid homomorphism given for each state q in Q by $h[\mathcal{A}](q, 1) = \iota(q)$ and $h[\mathcal{A}](q, 2) = \tau(q)$, and for each transition (p, a, q) in T by $h[\mathcal{A}](p, a, q) = a$. If $(h[\mathcal{A}](w) \mid w \in \|\mathrm{temp}(\mathcal{A})\|)$ is in $\mathcal{F}$, we shall write

$$\|\mathcal{A}\| := \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w) \tag{1}$$

and call the semiring element $\|\mathcal{A}\|$ the *behaviour* of $\mathcal{A}$. Otherwise we shall say that the behaviour $\|\mathcal{A}\|$ is undefined and write $\|\mathcal{A}\| = \bot$.

Remark 3.3. Note that we use the same notation $\|\cdot\|$ for languages recognised by template automata and for elements realised by automata over summation semirings. This presents a notational collision, as each template automaton is at the same time a finite automaton over a summation semiring of formal languages. However, it is easy to establish soundness of the definition by noting that the language $\|\mathrm{temp}(\mathcal{A})\|$ remains unchanged under the latter interpretation as well, i.e.,

$$\bigcup_{w \in \|\mathrm{temp}(\mathrm{temp}(\mathcal{A}))\|} h[\mathrm{temp}(\mathcal{A})](w) = \|\mathrm{temp}(\mathcal{A})\|$$

for $\|\mathrm{temp}(\mathcal{A})\|$ and $\|\mathrm{temp}(\mathrm{temp}(\mathcal{A}))\|$ interpreted as in the paragraph *above* Definition 3.2.

Remark 3.4. As explained in Example 2.2, each (countably) complete semiring is a summation semiring, in which all finite or (countably) infinite families of elements are summable. The behaviour $\|\mathcal{A}\|$ of a finite automaton $\mathcal{A}$ over such semiring, which is given by a finite or countably infinite sum (1), is thus always well defined. Let us now observe that this definition coincides with the usual definition of behaviour for automata over complete semirings [11]. We shall do so by providing a characterisation of behaviours of automata over *summation semirings*, which, when instantiated to complete semirings, yields one of the usual definitions of behaviour in this context.

A *run* of a finite S'-automaton $\mathcal{A} = (Q, \iota, T, \tau)$ over a summation semiring $(S, \mathcal{F}, \Phi)$ is given by a word $\gamma = q_0 a_1 q_1 \dots a_n q_n$ over a possibly infinite alphabet $Q \cup S'$, with n in $\mathbb{N}$, $q_0, \dots, q_n$ in Q, and $a_1, \dots, a_n$ in S', such that (q_{i-1}, a_i, q_i) is in T for $i = 1, \dots, n$. Each run $\gamma = q_0 a_1 q_1 \dots a_n q_n$ of $\mathcal{A}$ defines an element $\|\gamma\| := a_1 \dots a_n$ of S; moreover, let $\sigma(\gamma) := q_0$ be the *source* of γ and $\nu(\gamma) := q_n$ the *destination* of γ. Let us denote by $R(\mathcal{A})$ the set of all runs of $\mathcal{A}$. We will show that $\|\mathcal{A}\|$ can also be given by the sum

$$\sum_{\gamma \in R(\mathcal{A})} \iota(\sigma(\gamma)) \|\gamma\| \tau(\nu(\gamma)),$$

which exists if and only if $\|\mathcal{A}\|$ is defined. To show consistency of this alternative definition with the definition via (1), we have to prove that $(\iota(\sigma(\gamma))\|\gamma\|\tau(\nu(\gamma)) \mid \gamma \in R(\mathcal{A}))$ is in $\mathcal{F}$ if and only if $(h[\mathcal{A}](w) \mid w \in \|\mathrm{temp}(\mathcal{A})\|)$ is, while in that case

$$\sum_{\gamma \in R(\mathcal{A})} \iota(\sigma(\gamma)) \|\gamma\| \tau(\nu(\gamma)) = \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w).$$

However, the mapping $f_1 \colon R(\mathcal{A}) \to R(\mathrm{temp}(\mathcal{A}))$ defined for all runs $\gamma = q_0 a_1 q_1 \dots q_{n-1} a_n q_n$ in $R(\mathcal{A})$ by

$$f_1(\gamma) = q_0 (q_0, a_1, q_1) q_1 \dots q_{n-1} (q_{n-1}, a_n, q_n) q_n$$

is clearly a bijection. The same property is also obviously true for $f_2 \colon R(\mathrm{temp}(\mathcal{A})) \to \|\mathrm{temp}(\mathcal{A})\|$ given for all $\gamma' = q_0 (q_0, a_1, q_1) q_1 \dots q_{n-1} (q_{n-1}, a_n, q_n) q_n$ in $R(\mathrm{temp}(\mathcal{A}))$ by

$$f_2(\gamma') = (q_0, 1)(q_0, a_1, q_1)(q_1, a_2, q_2) \dots (q_{n-1}, a_n, q_n)(q_n, 2).$$

The mapping $f \colon R(\mathcal{A}) \to \|\mathrm{temp}(\mathcal{A})\|$, defined by $f := f_2 \circ f_1$, is therefore a bijection as well. For each $\gamma = q_0 a_1 q_1 \dots q_{n-1} a_n q_n$ in $R(\mathcal{A})$, we have

$$f(\gamma) = (q_0, 1)(q_0, a_1, q_1)(q_1, a_2, q_2) \dots (q_{n-1}, a_n, q_n)(q_n, 2),$$

so that

$$h[\mathcal{A}](f(\gamma)) = \iota(q_0) a_1 \dots a_n \tau(q_n) = \iota(\sigma(\gamma)) \|\gamma\| \tau(\nu(\gamma)).$$

Moreover, f being a bijection implies that $\|\mathrm{temp}(\mathcal{A})\|$ can be expressed as a *disjoint* union

$$\|\mathrm{temp}(\mathcal{A})\| = \bigcup_{\gamma \in R(\mathcal{A})} \{f(\gamma)\}.$$

It thus follows by the conditions (*i*) and (*ii*) of Definition 2.1 that the family $(h[\mathcal{A}](w) \mid w \in \|\mathrm{temp}(\mathcal{A})\|)$ is in $\mathcal{F}$ if and only if $(\iota(\sigma(\gamma))\|\gamma\|\tau(\nu(\gamma)) \mid \gamma \in R(\mathcal{A})) = (h[\mathcal{A}](f(\gamma)) \mid \gamma \in R(\mathcal{A}))$ is in $\mathcal{F}$ and that

$$\sum_{\gamma \in R(\mathcal{A})} \iota(\sigma(\gamma)) \|\gamma\| \tau(\nu(\gamma)) = \sum_{\gamma \in R(\mathcal{A})} h[\mathcal{A}](f(\gamma)) = \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w)$$

whenever this is the case, proving equivalence of both definitions.

Remark 3.5. It has been mentioned in Example 2.3 that each semiring of power series can be seen as a summation semiring, in which all locally finite families of series are summable. Finite automata over semirings of formal power series are usually called *weighted automata* [7]. It is well known that if a weighted automaton $\mathcal{A}$ is *proper* [22, 23] (a proper series is "read" on each transition; cf. Example 4.2), then the behaviour of $\mathcal{A}$ can be expressed as a sum over a locally finite family of power series.

More precisely, given a semiring S and an alphabet Σ, let $S\langle \Sigma \cup \{\varepsilon\} \rangle$ be the set of all power series r in $S\langle\langle \Sigma^* \rangle\rangle$ such that $(r, w) \neq 0$ for $w \in \Sigma^*$ implies that w is in $\Sigma \cup \{\varepsilon\}$. A weighted automaton can then be identified with a finite $S\langle \Sigma \cup \{\varepsilon\} \rangle$-automaton $\mathcal{A} = (Q, \iota, T, \tau)$ over the summation semiring $S\langle\langle \Sigma^* \rangle\rangle$ with locally finite families of series being summable.[3] A *proper* weighted automaton is then a weighted automaton such that $(r, \varepsilon) = 0$ for each (p, r, q) in T.

The behaviour $\|\mathcal{A}\|$ of a proper weighted automaton $\mathcal{A}$ is often defined via a possibly infinite sum over runs, as in Remark 3.4. The family summed over there is known to be locally finite, and the sum is thus well-defined for every semiring S.[4] Using the same argumentation as in Remark 3.4, it can be proved that this sum is equivalent to (1).

[3]One also usually supposes that $(\iota(q), c) = (\tau(q), c) = 0$ for each q in Q and c in Σ, so that the initial and terminal weights can be identified with elements of S.

[4]Using the fact that $S\langle \Sigma \cup \{\varepsilon\} \rangle$ is closed under addition and distributivity, it is easy to see that it is enough to have at most one transition (p, r, q) for each p, q in Q. Alternatively, one may decompose such transition into several transitions (p, r, q) such that $(r, c) \neq 0$ for *precisely one* c in Σ. "Restrictions" like these are commonly adopted in the theory of weighted automata.

The observations made in the two remarks above imply that the class of finite automata over summation semirings comprises both all automata over complete semirings and all proper automata over power series semirings (that is, all proper weighted automata).

Before going any further, let us give an example demonstrating the evaluation of behaviour of finite automata over summation semirings from their template languages.

Example 3.6. Let $\Sigma = \{a, b\}$ and consider the summation semiring of power series $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ with sums over locally finite families of series, where $\mathbb{N}$ is equipped with standard addition and multiplication. Let $\mathcal{A} = (Q, \iota, T, \tau)$ be the finite automaton over $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ depicted in Figure 1.

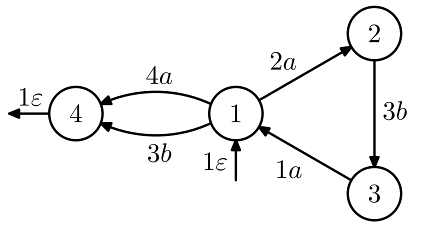

Figure 1: The finite automaton $\mathcal{A}$ over the summation semiring $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$.

The template automaton $\mathrm{temp}(\mathcal{A}) = (Q, \iota', T', \tau')$ differs from $\mathcal{A}$ in that transitions $(1, 2a, 2)$, $(2, 3b, 3)$, $(3, 1a, 1)$, $(1, 4a, 4)$, and $(1, 3b, 4)$ are respectively replaced by $(1,(1, 2a, 2), 2)$, $(2,(2, 3b, 3), 3)$, $(3,(3, 1a, 1), 1)$, $(1,(1, 4a, 4), 4)$, and $(1,(1, 3b, 4), 4)$, and in that each state $q = 1, \ldots, 4$ has $\iota'(q) = (q, 1)$ and $\tau'(q) = (q, 2)$. The language $\|\mathrm{temp}(\mathcal{A})\|$ is therefore given as follows:

$$\|\mathrm{temp}(\mathcal{A})\| = \bigcup_{p,q \in Q} (p, 1) L_{p,q} (q, 2),$$

where

$$\begin{aligned}
L_{1,1} &= ((1, 2a, 2)(2, 3b, 3)(3, 1a, 1))^*, & L_{3,1} &= (3, 1a, 1)L_{1,1},\\
L_{1,2} &= ((1, 2a, 2)(2, 3b, 3)(3, 1a, 1))^*(1, 2a, 2), & L_{3,2} &= (3, 1a, 1)L_{1,2},\\
L_{1,3} &= ((1, 2a, 2)(2, 3b, 3)(3, 1a, 1))^*(1, 2a, 2)(2, 3b, 3), & L_{3,3} &= ((3, 1a, 1)(1, 2a, 2)(2, 3b, 3))^*,\\
L_{1,4} &= ((1, 2a, 2)(2, 3b, 3)(3, 1a, 1))^*\{(1, 4a, 4),(1, 3b, 4)\}, & L_{3,4} &= (3, 1a, 1)L_{1,4},\\
L_{2,1} &= (2, 3b, 3)(3, 1a, 1)L_{1,1}, & L_{4,1} &= \emptyset,\\
L_{2,2} &= ((2, 3b, 3)(3, 1a, 1)(1, 2a, 2))^*, & L_{4,2} &= \emptyset,\\
L_{2,3} &= ((2, 3b, 3)(3, 1a, 1)(1, 2a, 2))^*(2, 3b, 3), & L_{4,3} &= \emptyset,\\
L_{2,4} &= (2, 3b, 3)(3, 1a, 1)L_{1,4}, & L_{4,4} &= \{\varepsilon\}.
\end{aligned}$$

The homomorphism $h[\mathcal{A}]\colon ((Q \times \{1, 2\}) \cup T)^* \to (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \cdot)$ is given by

$$\begin{aligned}
h[\mathcal{A}]((1, 2a, 2)) &= 2a, & h[\mathcal{A}]((1, 1)) &= 1\varepsilon, & h[\mathcal{A}]((1, 2)) &= 0,\\
h[\mathcal{A}]((2, 3b, 3)) &= 3b, & h[\mathcal{A}]((2, 1)) &= 0, & h[\mathcal{A}]((2, 2)) &= 0,\\
h[\mathcal{A}]((3, 1a, 1)) &= 1a, & h[\mathcal{A}]((3, 1)) &= 0, & h[\mathcal{A}]((3, 2)) &= 0,\\
h[\mathcal{A}]((1, 4a, 4)) &= 4a, & h[\mathcal{A}]((4, 1)) &= 0, & h[\mathcal{A}]((4, 2)) &= 1\varepsilon.\\
h[\mathcal{A}]((1, 3b, 4)) &= 3b,
\end{aligned}$$

As a consequence, we obtain the expected behaviour

$$\|\mathcal{A}\| = \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w) = \sum_{w \in (1,1)L_{1,4}(4,2)} h[\mathcal{A}](w) + \sum_{\substack{p,q \in Q \\ p \neq 1 \,\vee\, q \neq 4}} \sum_{w \in (p,1)L_{p,q}(q,2)} h[\mathcal{A}](w) =$$

$$= \left(\sum_{n \in \mathbb{N}} h[\mathcal{A}]\left((1,1)((1,2a,2)(2,3b,3)(3,1a,1))^n(1,4a,4)(4,2)\right) + \right.$$

$$\left. + \sum_{n \in \mathbb{N}} h[\mathcal{A}]\left((1,1)((1,2a,2)(2,3b,3)(3,1a,1))^n(1,3b,4)(4,2)\right) \right) + 0 =$$

$$= \sum_{n \in \mathbb{N}} \left((4 \cdot 6^n)(aba)^n a + (3 \cdot 6^n)(aba)^n b\right).$$

The following lemma states that if a family of homomorphic images of words in a *rational* language is summable in some summation semiring S, then the sum over this family is a behaviour of some finite automaton over S. In other words, elements realised by finite automata over S can not only be characterised as sums of homomorphic images of words from rational languages recognised by template automata, but also as sums of homomorphic images of words from arbitrary rational languages.

Lemma 3.7. *Let Σ be an alphabet, $L \subseteq \Sigma^*$ a rational language, $(S, \mathcal{F}, \Phi)$ a summation semiring, and $h\colon \Sigma^* \to (S, \cdot)$ a monoid homomorphism. If $(h(w) \mid w \in L)$ is in $\mathcal{F}$, then the semiring element*

$$\sum_{w \in L} h(w)$$

equals the behaviour $\|\mathcal{A}\|$ of some finite S'-automaton $\mathcal{A}$ over S with $S' = h(\Sigma) \cup \{0,1\}$.

Proof. As L is rational, it is recognised by some *unambiguous* finite $(2_1^{\Sigma} \cup \{\emptyset, \{\varepsilon\}\})$-automaton $\mathcal{B} = (Q, \iota, T, \tau)$ over 2^{Σ^*} such that $\iota(q)$ and $\tau(q)$ are either $\emptyset$ or $\{\varepsilon\}$ for all q in Q and a is in 2_1^{Σ} (identified with Σ) whenever (p,a,q) is in T for some p,q in Q. Without loss of generality, let us also assume that there is *at most one* c in Σ for each p,q in Q such that (p,c,q) is in T – this normal form can be easily obtained by maintaining a copy of each state for each last read symbol.

Let us now construct a finite S'-automaton $\mathcal{A}$ over S as follows: $\mathcal{A} = (Q, \iota', T', \tau')$, where

$$\iota'(q) = \begin{cases} 1 & \text{if } \iota(q) = \{\varepsilon\} \\ 0 & \text{if } \iota(q) = \emptyset \end{cases}$$

and

$$\tau'(q) = \begin{cases} 1 & \text{if } \tau(q) = \{\varepsilon\} \\ 0 & \text{if } \tau(q) = \emptyset \end{cases}$$

for each q in Q, and where $T' = \{(p, h(c), q) \mid (p,c,q) \in T\}$. Let $\chi(\mathcal{A})$ be the language of all x in $(T')^*$ such that $(p,1)x(q,2)$ is in $\|\mathrm{temp}(\mathcal{A})\|$ for some p,q in Q such that $\iota'(p) = 1$ and $\tau'(q) = 1$; let $\chi'(\mathcal{A})$ be the language of all $(p,1)x(q,2)$ in $\|\mathrm{temp}(\mathcal{A})\|$ such that $\iota'(p) = 0$ or $\tau'(q) = 0$. It is then clear that if $f\colon (T')^* \to \Sigma^*$ is a homomorphism defined by $f(p,h(c),q) = c$ for all p,q in Q and c in Σ such that (p,c,q) is in T (there is at most one such c for each p,q by our earlier assumption; f is thus well-defined), then

$$L = \bigcup_{x \in \chi(\mathcal{A})} \{f(x)\}. \tag{2}$$

Moreover, it is clear that $h[\mathcal{A}](x) = h(f(x))$ for all x in $\chi(\mathcal{A})$. As $\mathcal{B}$ is unambiguous, the union in (2) is disjoint. It then follows by conditions (*i*) to (*iv*) of Definition 2.1 that the families $(h[\mathcal{A}](x) \mid x \in \chi(\mathcal{A}))$ and $(h[\mathcal{A}](x) \mid x \in \|\mathrm{temp}(\mathcal{A})\|)$ are summable whenever $(h(w) \mid w \in L)$ is, in which case

$$\|\mathcal{A}\| = \sum_{x \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](x) = \sum_{x \in \chi(\mathcal{A})} h[\mathcal{A}](x) + \sum_{x \in \chi'(\mathcal{A})} 0 = \sum_{x \in \chi(\mathcal{A})} h[\mathcal{A}](x) = \sum_{x \in \chi(\mathcal{A})} h(f(x)) = \sum_{w \in L} h(w).$$

This proves the statement of the lemma. □

Remark 3.8. Note that the automaton $\mathcal{A} = (Q, \iota', T', \tau')$ constructed in the proof of Lemma 3.7 has a property that $\iota'(q)$ is either 0 or 1 for each state q in Q and the same property holds for $\tau'(q)$. As the rational language L from the statement of Lemma 3.7 can be chosen as $L = \|\mathrm{temp}(\mathcal{C})\|$ for each finite S'-automaton $\mathcal{C}$ over a summation semiring S, this proves that each finite S'-automaton over S can be transformed into a normal form, in which both the initial weight and the terminal weight of each state q is in $\{0, 1\}$.

Moreover, observe that this property has been established without actually doing the construction, utilising the well-known fact that it is possible to construct a finite automaton $\mathcal{B} = (Q, \iota, T, \tau)$ for the language L such that $\iota(q)$ and $\tau(q)$ are in $\{\emptyset, \{\varepsilon\}\}$ for each state q in Q. Stronger normal forms for finite S'-automata over a summation semiring S can be obtained in a similar way as well. For instance, by assuming that $\mathcal{B}$ is *normalised* [22, 23] – i.e., that it has a single initial state p, which is not a destination of any transition, and a single terminal state $q \neq p$, which is not a source of any transition – it is possible to "lift" this property to automata over summation semirings. That is: each finite S'-automaton over a summation semiring S is equivalent to some finite S'-automaton $\mathcal{A} = (Q, \iota', T', \tau')$ over S, for which there are p, q in Q such that $p \neq q$, $\iota'(p) = \tau'(q) = 1$, $\iota'(p') = 0$ for each p' in Q different from p, $\tau'(q') = 0$ for each q' in Q different from q, and T' is a finite subset of $(Q - \{q\}) \times S' \times (Q - \{p\})$.

One of the widely used approaches to defining series realised by weighted finite automata or elements realised by finite automata over complete semirings utilises the star of a *transition matrix* corresponding to an automaton [7, 11, 22, 23]. In the setting of summation semirings, it may happen that the behaviour of a finite automaton is defined although some entries of the star of its transition matrix are not. Conversely, a sum by which the behaviour of a finite automaton is given might be undefined even if the star of its transition matrix is defined.[5] This means that the definition of a realised semiring element via matrices is not a proper equivalent of Definition 3.2.

Nevertheless, we shall now prove that both definitions are equivalent whenever both the behaviour and the star of a transition matrix are defined. In other words, the approach based on matrices results in the same notion of realised semiring elements, but these are well defined for a slightly different class of automata. On the other hand, it is easy to see that this class still contains all finite automata over complete semirings and all proper weighted automata.

Let us first define the *star* A^* of a square matrix A over a summation semiring. This is defined to be, if it exists, the *elementwise* sum of all nonnegative integer powers of A.

Definition 3.9. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, $n \geq 1$ in $\mathbb{N}$, and $A = (a_{i,j})_{n \times n}$ a square matrix over S. For each t in $\mathbb{N}$, let

$$A^t = \begin{pmatrix} a_{1,1}[t] & a_{1,2}[t] & \dots & a_{1,n}[t] \\ a_{2,1}[t] & a_{2,2}[t] & \dots & a_{2,n}[t] \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}[t] & a_{n,2}[t] & \dots & a_{n,n}[t] \end{pmatrix}.$$

If $(a_{i,j}[t] \mid t \in \mathbb{N})$ is in $\mathcal{F}$ for $i, j = 1, \dots, n$, we shall define the *star* A^* of A by

$$A^* = \begin{pmatrix} \sum_{t \in \mathbb{N}} a_{1,1}[t] & \sum_{t \in \mathbb{N}} a_{1,2}[t] & \dots & \sum_{t \in \mathbb{N}} a_{1,n}[t] \\ \sum_{t \in \mathbb{N}} a_{2,1}[t] & \sum_{t \in \mathbb{N}} a_{2,2}[t] & \dots & \sum_{t \in \mathbb{N}} a_{2,n}[t] \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t \in \mathbb{N}} a_{n,1}[t] & \sum_{t \in \mathbb{N}} a_{n,2}[t] & \dots & \sum_{t \in \mathbb{N}} a_{n,n}[t] \end{pmatrix}.$$

Otherwise we shall say that the star A^* of A is undefined.

Without loss of generality, we shall now confine ourselves to automata with state sets $\{1, \dots, n\}$ for some $n \geq 1$ in $\mathbb{N}$. This is no real restriction, as it essentially corresponds to fixing a linear order on the set of states and identifying each state with its position in this order. However, note that we shall do this for notational convenience only, as it is possible to use matrices and vectors indexed by the set of states Q instead [11].

[5] In fact, this latter inconsistency is not so unusual as it may seem – it may already happen for cycle-free weighted automata over arbitrary coefficient semirings, as defined in [11]. We shall return to this point in Section 8.

Definition 3.10. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, $n \geq 1$ in $\mathbb{N}$, and $\mathcal{A} = (\{1, \ldots, n\}, \iota, T, \tau)$ a finite S'-automaton over S. For each p, q in Q, let $\sigma(p, q)$ be defined by the *finite* sum

$$\sigma(p,q) = \sum_{\substack{a \in S' \\ (p,a,q) \in T}} a.$$

The *initial vector* $I_{\mathcal{A}}$ and the *terminal vector* $F_{\mathcal{A}}$ corresponding to $\mathcal{A}$ are then defined by

$$I_{\mathcal{A}} = (\iota(1), \iota(2), \ldots, \iota(n)) \qquad \text{and} \qquad F_{\mathcal{A}} = (\tau(1), \tau(2), \ldots, \tau(n))^T,$$

and the *transition matrix* $\Delta_{\mathcal{A}}$ of $\mathcal{A}$ is defined by

$$\Delta_{\mathcal{A}} = \begin{pmatrix} \sigma(1,1) & \sigma(1,2) & \ldots & \sigma(1,n) \\ \sigma(2,1) & \sigma(2,2) & \ldots & \sigma(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(n,1) & \sigma(n,2) & \ldots & \sigma(n,n) \end{pmatrix}.$$

If $\mathcal{A}$ is clear from the context, we may write I, Δ, and F instead of $I_{\mathcal{A}}$, $\Delta_{\mathcal{A}}$, and $F_{\mathcal{A}}$.

We are now prepared to prove that if $\mathcal{A}$ is an automaton with transition matrix Δ such that Δ^* and $\|\mathcal{A}\|$ are both defined, then $I\Delta^*F$ coincides with $\|\mathcal{A}\|$.

Theorem 3.11. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and $\mathcal{A} = (\{1, \ldots, n\}, \iota, T, \tau)$ a finite S'-automaton over S for some $n \geq 1$ in $\mathbb{N}$. Let I be the initial vector, Δ the transition matrix, and F the terminal vector of $\mathcal{A}$. If $\|\mathcal{A}\|$ and Δ^* are both defined, then*

$$\|\mathcal{A}\| = I\Delta^*F.$$

Proof. Let E be the transition matrix of $\mathrm{temp}(\mathcal{A})$. Let $E^t = (e_{i,j}[t])_{n \times n}$ for each t in $\mathbb{N}$. It is easy to prove by induction that if $\Delta^t = (d_{i,j}[t])_{n \times n}$ for each t in $\mathbb{N}$, then $d_{i,j}[t]$ is given by the *finite* sum

$$d_{i,j}[t] = \sum_{x \in e_{i,j}[t]} h[\mathcal{A}](x)$$

for $i, j = 1, \ldots, n$ and each t in $\mathbb{N}$. It then follows by the obvious unambiguity of the template automaton $\mathrm{temp}(\mathcal{A})$ that $\|\mathrm{temp}(\mathcal{A})\|$ can be expressed by the *disjoint* union

$$\|\mathrm{temp}(\mathcal{A})\| = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \bigcup_{t \in \mathbb{N}} (i,1) e_{i,j}[t] (j,2).$$

As a result, it follows by conditions (i) to (iv) of Definition 2.1 that

$$\begin{aligned} \|\mathcal{A}\| = \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t \in \mathbb{N}} \sum_{x \in e_{i,j}[t]} h[\mathcal{A}]((i,1)x(j,2)) = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t \in \mathbb{N}} h[\mathcal{A}](i,1) \left(\sum_{x \in e_{i,j}[t]} h[\mathcal{A}](x) \right) h[\mathcal{A}](j,2) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t \in \mathbb{N}} \iota(i) d_{i,j}[t] \tau(j) = \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \iota(i) \left(\sum_{t \in \mathbb{N}} d_{i,j}[t] \right) \tau(j) = I\Delta^*F, \end{aligned}$$

all sums being over summable families. The theorem is proved. □

4. Summation Semirings and Partial Conway Semirings

Bloom, Ésik, and Kuich [2, 12] have developed a theory of finite automata and rational semiring elements in the framework of partial Conway semirings over an ideal (see Section 2 for the definition of such semirings). Similarly to finite automata over summation semirings, the notion of finite automata over partial Conway semirings unifies the usual notions of finite automata over complete semirings and proper automata over arbitrary power series semirings (and cycle-free automata are incorporated as well).

Let us first recall the basic definitions. An *ideal* [2, 12, 14] in a semiring S is a nonempty subset I of S such that $a + b$ is in I whenever a, b are in I and such that $a \cdot x$ and $x \cdot a$ are in I whenever a is in I and x is in S.

Let S be a semiring and I an ideal in S. The semiring S is called a *partial Conway semiring* [2, 12] over I if it is equipped with a unary operation $^*\colon I \to S$ such that the *sum star identity*

$$(a + b)^* = a^* \cdot (b \cdot a^*)^*$$

holds for all a, b in I and the *product star identity*

$$(a \cdot b)^* = 1 + a \cdot (b \cdot a)^* \cdot b$$

holds for all a, b in S such that a is in I or b is in I. A partial Conway semiring S over the ideal S is called a *Conway semiring* [2, 6, 12].

Example 4.1. It is well known [6] that if a star operation is defined for all a in a complete semiring S by

$$a^* = \sum_{i \in \mathbb{N}} a^i,$$

then the sum star identity and the product star identity hold for all a, b in S. Hence, each complete semiring is a partial Conway semiring (a Conway semiring, in fact).

Example 4.2. A formal power series r in $S\langle\langle \Sigma^* \rangle\rangle$ (cf. Example 2.3) is said to be *proper* if $(r, \varepsilon) = 0$. It is proved in [2, 12] that if S is a partial Conway semiring over some ideal I and Σ is an alphabet, then $S\langle\langle \Sigma^* \rangle\rangle$ is a partial Conway semiring over the ideal $\{r \in S\langle\langle \Sigma^* \rangle\rangle \mid (r, \varepsilon) \in I\}$. As an important special case, consider $I = \{0\}$: the set $S\langle\langle \Sigma^+ \rangle\rangle$ of all proper series in $S\langle\langle \Sigma^* \rangle\rangle$ is an ideal. Moreover, it is easy to see [2, 6, 12] that if r is in $S\langle\langle \Sigma^+ \rangle\rangle$, then the family of series $(r^i \mid i \in \mathbb{N})$ is locally finite, and thus can be summed as in Example 2.3. The star r^* of a proper series r can thus be defined as a sum of this locally finite family. It is then possible to prove [2, 12] that the sum star identity holds for each pair of proper power series and that the product star identity holds for each pair of power series such that at least one of them is proper. This implies that each semiring of power series is a partial Conway semiring over the ideal of proper series.

In what follows, we shall show that the two alternative unifying approaches provided by summation semirings and partial Conway semirings have incomparable scopes. This means that there exists a summation semiring that is not a partial Conway semiring over an ideal and conversely, there is a partial Conway semiring that is not a summation semiring.

Remark 4.3. The above claim should be made more precise in order to rule out possible trivial interpretations. To this end, we shall say that a summation semiring $(S, \mathcal{F}, \Phi)$ *is a partial Conway semiring over an ideal* if the set I of all a in S such that $(a^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$ forms an ideal in S and if the sum star and product star identities hold for a star operation defined for all a in I by

$$a^* = \sum_{i \in \mathbb{N}} a^i.$$

Conversely, we shall say that a partial Conway semiring S over an ideal I *is a summation semiring*, if it is possible to find $\mathcal{F}$ and Φ so that $(S, \mathcal{F}, \Phi)$ is a summation semiring, $(a^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$ for all a in I, and

$$a^* = \sum_{i \in \mathbb{N}} a^i$$

holds for each a in I. We shall always conform to this convention in what follows.

It has to be noted that although the following two propositions show their incomparability, the settings most relevant from the viewpoint of automata theory seem to be within the scope of *both* summation semirings and partial Conway semirings.

Proposition 4.4. *There is a summation semiring that is not a partial Conway semiring over an ideal.*

Proof. Consider the semiring $(\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$ of nonnegative real numbers with usual operations of addition and multiplication, which can be turned into a summation semiring $(\mathbb{R}_{\geq 0}, \mathcal{F}, \Phi)$, where $\mathcal{F}$ consists of all finite families $(a_i \mid i \in \{i_1, \ldots, i_n\})$ in $\mathcal{F}(\mathbb{R}_{\geq 0})$ and all countably infinite families $(a_i \mid i \in \{i_1, i_2, \ldots\})$ in $\mathcal{F}(\mathbb{R}_{\geq 0})$ such that the infinite series

$$\sum_{j=1}^{\infty} a_{i_j}$$

converges. This convergence is necessarily absolute – the sum of a series and the property of a family being in $\mathcal{F}$ thus do not depend on the order of elements. Hence, it is possible to define the function Φ for all finite $I = \{i_1, \ldots, i_n\}$ and all $(a_i \mid i \in I)$ in $\mathcal{F}$ by

$$\Phi(a_i \mid i \in I) = \sum_{i \in I} a_i = \sum_{j=1}^{n} a_{i_j}$$

and for all countably infinite $I = \{i_1, i_2, \ldots\}$ and all $(a_i \mid i \in I)$ in $\mathcal{F}$ by

$$\Phi(a_i \mid i \in I) = \sum_{i \in I} a_i = \sum_{j=1}^{\infty} a_{i_j}.$$

It is easy to check that the conditions (*i*) to (*iv*) of Definition 2.1 are indeed satisfied.

On the other hand, this summation semiring is not a partial Conway semiring over an ideal, as $(a^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$ if and only if $0 \leq a < 1$. However, the real interval $[0, 1)$ is not an ideal in $\mathbb{R}_{\geq 0}$. ☐

Proposition 4.5. *There is a partial Conway semiring over an ideal that is not a summation semiring.*

Proof. Consider the semiring $(\text{Rat } a^*, \cup, \cdot, \emptyset, \{\varepsilon\})$ of rational languages over the unary alphabet $\{a\}$ together with the standard operations of union and concatenation. This is a Conway semiring – that is, a partial Conway semiring over $\text{Rat } a^*$ – for the star of a language defined in the usual way.

Suppose for contradiction that $(\text{Rat } a^*, \cup, \cdot, \emptyset, \{\varepsilon\})$ forms a summation semiring $(\text{Rat } a^*, \mathcal{F}, \Phi)$ for some $\mathcal{F}$ and Φ, so that $(L^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$ and

$$\Phi(L^i \mid i \in \mathbb{N}) = \sum_{i \in \mathbb{N}} L^i = L^*$$

for all L in $\text{Rat } a^*$.[6] In particular, $(\{a^i\} \mid i \in \mathbb{N})$ is in $\mathcal{F}$ and

$$a^* = \sum_{i \in \mathbb{N}} \{a^i\}.$$

It then follows by condition (*ii*) of Definition 2.1 that $(\{a^i\} \mid i \in I)$ is in $\mathcal{F}$ for $I = \{n^2 \mid n \in \mathbb{N}\}$. Let

$$L = \sum_{i \in I} \{a^i\}. \tag{3}$$

[6] Note that we denote the infinite summation in $(\text{Rat } a^*, \mathcal{F}, \Phi)$ by $\sum_{i \in \mathbb{N}} L^i$, as opposed to $\bigcup_{i \in \mathbb{N}} L^i$. The reason for this is that it is not *a priori* clear that the infinite summation has to coincide with the usually defined infinite union.

For n in I, the word a^n has to be in L, as conditions (ii) and (i) of Definition 2.1 imply that

$$L = \sum_{i \in I} \{a^i\} = \left(\sum_{i \in I - \{n\}} \{a^i\} \right) \cup \left(\sum_{i \in \{n\}} \{a^i\} \right) = L' \cup \{a^n\}$$

for some language L'. On the other hand, if n is not in I, it follows by iteration of the product star identity that

$$a^* = \{a^0\} \cup \{a^1\} \cup \ldots \cup \{a^{n-1}\} \cup \{a^n\} \cup a^{n+1} \cdot a^*,$$

from which we obtain, by conditions (i) to (iv) of Definition 2.1,

$$\begin{aligned} a^* - \{a^n\} &= \{a^0\} \cup \{a^1\} \cup \ldots \cup \{a^{n-1}\} \cup a^{n+1} \cdot a^* = \\ &= \{a^0\} \cup \{a^1\} \cup \ldots \cup \{a^{n-1}\} \cup a^{n+1} \cdot \sum_{i \in \mathbb{N}} \{a^i\} = \sum_{i \in \mathbb{N} - \{n\}} \{a^i\}. \end{aligned}$$

Now, as $I \subseteq \mathbb{N} - \{n\}$, condition (ii) of Definition 2.1 gives us

$$a^* - \{a^n\} = \left(\sum_{i \in I} \{a^i\} \right) \cup \left(\sum_{i \in \mathbb{N} - \{n\} - I} \{a^i\} \right) = L \cup L''$$

for some language L'', from which it follows that a^n is not in L. As a result, the word a^n is in L if and only if n is in I – that is,

$$L = \{a^{n^2} \mid n \in \mathbb{N}\}.$$

However, the language L is not rational, meaning that (3) cannot hold – a contradiction. □

5. Rational Expressions over Summation Semirings

We shall now define rational expressions over summation semirings and prove their equivalence with finite automata, given that their semantics is suitably defined. The syntax of rational expressions shall be defined in a standard way, as for any other class of semirings.

Definition 5.1. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. A *rational S'-expression* over S is a word from a language defined inductively as follows:

1. If a is in S', then a is a rational S'-expression over S called an *atomic expression* or an *atom*.
2. If E and E' are rational S'-expressions over S, then $(\mathsf{E} + \mathsf{E}')$ and $(\mathsf{E} \cdot \mathsf{E}')$ are rational S'-expressions over S as well.
3. If E is a rational S'-expression over S, then (E^*) is a rational S'-expression over S as well.
4. Nothing else is a rational S'-expression over S.

To specify the semantics, we shall associate what we shall call a *template expression* $\mathrm{temp}(\mathsf{E})$ to each rational expression E over a summation semiring. Roughly speaking, this shall be done by replacing each atom of the original expression with a distinct symbol from some alphabet and by adding some other symbols ensuring that the template expression is unambiguous. The atoms of a template expression will always be $\mathsf{c}_0, \ldots, \mathsf{c}_n$ for some n in $\mathbb{N}$ and symbols $c_0, \ldots, c_n$. However, in order to allow for a natural recursive definition of template expressions, we shall first define $\mathrm{temp}_k(\mathsf{E})$ for each k in $\mathbb{N}$, in which the atoms will be $\mathsf{c}_k, \ldots, \mathsf{c}_{k+n}$, and then define the template expression to be $\mathrm{temp}_0(\mathsf{E})$. The *semiring element denoted by a rational expression* E will then be defined in terms of the language $\|\mathrm{temp}(\mathsf{E})\|$ denoted by its template expression (under the usual semantics [22, 15]) and a suitable homomorphism.

Recall that if X is a set, then 2^X_1 denotes the set of all singleton subsets of X, while each $\{x\}$ in 2^X_1 is identified with x in X.

Definition 5.2. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. For all rational S'-expressions E over S and all k in $\mathbb{N}$, let the alphabet $\Sigma[\mathsf{E}, k]$, the homomorphism $h[\mathsf{E}, k] \colon \Sigma[\mathsf{E}, k]^* \to (S, \cdot)$, and the rational $(2_1^{\Sigma[\mathsf{E},k]} \cup \{\emptyset, \{\varepsilon\}\})$-expression $\mathrm{temp}_k(\mathsf{E})$ over $2^{\Sigma[\mathsf{E},k]^*}$ be defined inductively as follows:

1. For each a in S' and k in $\mathbb{N}$, let $\Sigma[\mathsf{a}, k] = \{c_k\}$, the homomorphism $h[\mathsf{a}, k] \colon \Sigma[\mathsf{a}, k]^* \to (S, \cdot)$ be given by $h[\mathsf{a}, k](c_k) = a$, and $\mathrm{temp}_k(\mathsf{a}) = \mathsf{c}_k$.
2. If E, E' are rational S'-expressions over S, k is in $\mathbb{N}$, and m is the number of elements of $\Sigma[\mathsf{E}, k]$, then $\Sigma[(\mathsf{E}+\mathsf{E}'), k] = \Sigma[(\mathsf{E}\cdot\mathsf{E}'), k] = \Sigma[\mathsf{E}, k] \cup \Sigma[\mathsf{E}', k+m]$, the homomorphism $h[(\mathsf{E}+\mathsf{E}'), k] = h[(\mathsf{E}\cdot\mathsf{E}'), k] =: h$ is defined by $h(c) = h[\mathsf{E}, k](c)$ for all c in $\Sigma[\mathsf{E}, k]$ and by $h(c) = h[\mathsf{E}', k+m](c)$ for all c in $\Sigma[\mathsf{E}', k+m]$, while $\mathrm{temp}_k((\mathsf{E} + \mathsf{E}')) = (\mathrm{temp}_k(\mathsf{E}) + \mathrm{temp}_{k+m}(\mathsf{E}'))$ and $\mathrm{temp}_k((\mathsf{E} \cdot \mathsf{E}')) = (\mathrm{temp}_k(\mathsf{E}) \cdot \mathrm{temp}_{k+m}(\mathsf{E}'))$.
3. If E is a rational S'-expression over S, k is in $\mathbb{N}$, and m is the number of elements of $\Sigma[\mathsf{E}, k]$, then $\Sigma[(\mathsf{E}^*), k] = \Sigma[\mathsf{E}, k] \cup \{c_{k+m}\}$, $h[(\mathsf{E}^*), k](c) = h[\mathsf{E}, k](c)$ for each c in $\Sigma[\mathsf{E}, k]$, $h[(\mathsf{E}^*), k](c_{k+m}) = 1$, and $\mathrm{temp}_k((\mathsf{E}^*)) = ((\mathrm{temp}_k(\mathsf{E}) \cdot \mathsf{c}_{k+m})^*)$.

Remark 5.3. The reason for introducing the symbol c_{k+m} in $\mathrm{temp}_k((\mathsf{E}^*))$ is to make the stars in template expressions unambiguous (this property will be used in the proof of Proposition 5.7). Actually, as union and concatenation are obviously unambiguous in template expressions, this implies that template expressions are always unambiguous rational expressions. One could get ambiguous template expressions if these symbols $\mathsf{c}_{k,m}$ were not used: for instance the outer star in $((\mathsf{c}_0^*)^*)$ is ambiguous.

Definition 5.4. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing both 0 and 1, and E a rational S'-expression over S. Let $\Sigma[\mathsf{E}] = \Sigma[\mathsf{E}, 0]$, $h[\mathsf{E}] = h[\mathsf{E}, 0]$, and the *template expression* $\mathrm{temp}(\mathsf{E})$ corresponding to E be defined by $\mathrm{temp}(\mathsf{E}) = \mathrm{temp}_0(\mathsf{E})$.[7] If $(h[\mathsf{E}](w) \mid w \in \|\mathrm{temp}(\mathsf{E})\|)$ is in $\mathcal{F}$, we shall write

$$\|\mathsf{E}\| := \sum_{w \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](w)$$

and call the semiring element $\|\mathsf{E}\|$ the *element denoted by* E. Otherwise we shall say that the element denoted by E is undefined and write $\|\mathsf{E}\| = \perp$.

Remark 5.5. Similarly as for behaviours of finite automata, we use the same notation $\|\cdot\|$ for languages denoted by template expressions and for elements denoted by rational expressions over summation semirings. Despite this notational collision, Definition 5.2 is sound, as it is easy to see that the language $\|\mathrm{temp}(\mathsf{E})\|$ denoted by the template expression $\mathrm{temp}(\mathsf{E})$ remains unchanged even if $\mathrm{temp}(\mathsf{E})$ is interpreted as a rational expression over a summation semiring of formal languages.

Example 5.6. Let us construct a rational expression E over the summation semiring $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$, with $\Sigma = \{a, b\}$ and sums over locally finite families of series, such that $\|\mathsf{E}\|$ equals the behaviour of the automaton given in Example 3.6, i.e.,

$$\sum_{n \in \mathbb{N}} ((4 \cdot 6^n)(aba)^n a + (3 \cdot 6^n)(aba)^n b)\,.$$

An obvious candidate is, if we remove some of the parentheses for clarity, the rational expression

$$\mathsf{E} = (6\mathsf{a} \cdot \mathsf{b} \cdot \mathsf{a})^* \cdot (4\mathsf{a} + 3\mathsf{b}).$$

The template expression corresponding to E is given by

$$\mathrm{temp}(\mathsf{E}) = (\mathsf{c}_0 \cdot \mathsf{c}_1 \cdot \mathsf{c}_2 \cdot \mathsf{c}_3)^* \cdot (\mathsf{c}_4 + \mathsf{c}_5),$$

so that

$$\|\mathrm{temp}(\mathsf{E})\| = (c_0 c_1 c_2 c_3)^* \{c_4, c_5\}.$$

[7] In what follows, $\|\mathrm{temp}(\mathsf{E})\|$ is the language denoted by $\mathrm{temp}(\mathsf{E})$ defined in the usual way. See also Remark 5.5.

Moreover, the homomorphism $h[\mathsf{E}]$ is given by $h[\mathsf{E}](c_0) = 6a$, $h[\mathsf{E}](c_1) = b$, $h[\mathsf{E}](c_2) = a$, $h[\mathsf{E}](c_3) = 1$, $h[\mathsf{E}](c_4) = 4a$, and $h[\mathsf{E}](c_5) = 3b$. As a result, we indeed obtain

$$\begin{aligned}\|\mathsf{E}\| &= \sum_{w \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](w) = \sum_{n \in \mathbb{N}} h[\mathsf{E}]\left((c_0c_1c_2c_3)^n c_4\right) + \sum_{n \in \mathbb{N}} h[\mathsf{E}]\left((c_0c_1c_2c_3)^n c_5\right) = \\ &= \sum_{n \in \mathbb{N}} \left((4 \cdot 6^n)(aba)^n a + (3 \cdot 6^n)(aba)^n b\right),\end{aligned}$$

which is the same series as in Example 3.6.

We shall now prove that the semantics of rational expressions over summation semirings, as introduced in Definition 5.2, behaves in a relatively reasonable way.

Proposition 5.7. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and E, E' rational S'-expressions over S such that $\|\mathsf{E}\| \neq \bot$ and $\|\mathsf{E}'\| \neq \bot$. Then $\|(\mathsf{E} + \mathsf{E}')\| = \|\mathsf{E}\| + \|\mathsf{E}'\|$ and $\|(\mathsf{E} \cdot \mathsf{E}')\| = \|\mathsf{E}\| \cdot \|\mathsf{E}'\|$. If moreover $\|(\mathsf{E}^*)\| \neq \bot$, then $(\|\mathsf{E}\|^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$ and*

$$\|(\mathsf{E}^*)\| = \sum_{i \in \mathbb{N}} \|\mathsf{E}\|^i.$$

Proof. Using conditions (ii) and (i) of Definition 2.1, it is easy to prove by structural induction that

$$\sum_{w \in \|\mathrm{temp}_k(\mathsf{F})\|} h[\mathsf{F}, k](w) = \sum_{w \in \|\mathrm{temp}(\mathsf{F})\|} h[\mathsf{F}](w) \tag{4}$$

holds for each rational expression F and each k in $\mathbb{N}$. Let m be the number of elements of $\Sigma[\mathsf{E}] = \Sigma[\mathsf{E}, 0]$. Then, by Definition 5.2,

$$\|\mathrm{temp}((\mathsf{E} + \mathsf{E}'))\| = \|(\mathrm{temp}_0(\mathsf{E}) + \mathrm{temp}_m(\mathsf{E}'))\| = \|\mathrm{temp}_0(\mathsf{E})\| \cup \|\mathrm{temp}_m(\mathsf{E}')\|,$$

with the union being disjoint.[8] Hence, it follows by conditions (i) and (iii) of Definition 2.1 and by (4) that $(h[(\mathsf{E} + \mathsf{E}')](w) \mid w \in \|\mathrm{temp}((\mathsf{E} + \mathsf{E}'))\|)$ is in $\mathcal{F}$ and

$$\begin{aligned}\|(\mathsf{E} + \mathsf{E}')\| &= \sum_{w \in \|\mathrm{temp}((\mathsf{E}+\mathsf{E}'))\|} h[(\mathsf{E} + \mathsf{E}')](w) = \sum_{w \in \|\mathrm{temp}_0(\mathsf{E})\|} h[(\mathsf{E} + \mathsf{E}')](w) + \sum_{w \in \|\mathrm{temp}_m(\mathsf{E}')\|} h[(\mathsf{E} + \mathsf{E}')](w) = \\ &= \sum_{w \in \|\mathrm{temp}_0(\mathsf{E})\|} h[\mathsf{E}, 0](w) + \sum_{w \in \|\mathrm{temp}_m(\mathsf{E}')\|} h[\mathsf{E}', m](w) = \\ &= \sum_{w \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](w) + \sum_{w \in \|\mathrm{temp}(\mathsf{E}')\|} h[\mathsf{E}'](w) = \|\mathsf{E}\| + \|\mathsf{E}'\|.\end{aligned}$$

Next,

$$\|\mathrm{temp}((\mathsf{E} \cdot \mathsf{E}'))\| = \|(\mathrm{temp}_0(\mathsf{E}) \cdot \mathrm{temp}_m(\mathsf{E}'))\| = \|\mathrm{temp}_0(\mathsf{E})\| \cdot \|\mathrm{temp}_m(\mathsf{E}')\|.$$

[8] This follows by the fact that $\Sigma[\mathsf{E}, 0]$ and $\Sigma[\mathsf{E}', m]$ are disjoint and that languages $\|\mathrm{temp}_k(\mathsf{F})\|$ are ε-free for all rational expressions F and all k in $\mathbb{N}$.

Thus, by conditions (ii), (i), and (iv) of Definition 2.1, by the fact that $\Sigma[\mathsf{E},0] \cap \Sigma[\mathsf{E}',m] = \emptyset$, and by (4),

$$\begin{aligned}
\|(\mathsf{E} \cdot \mathsf{E}')\| &= \sum_{w \in \|\mathrm{temp}((\mathsf{E}\cdot\mathsf{E}'))\|} h[(\mathsf{E} \cdot \mathsf{E}')](w) = \sum_{w \in \|\mathrm{temp}_0(\mathsf{E})\| \cdot \|\mathrm{temp}_m(\mathsf{E}')\|} h[(\mathsf{E} \cdot \mathsf{E}')](w) = \\
&= \sum_{(u,v) \in \|\mathrm{temp}_0(\mathsf{E})\| \times \|\mathrm{temp}_m(\mathsf{E}')\|} h[\mathsf{E},0](u) \cdot h[\mathsf{E}',m](v) = \\
&= \left(\sum_{u \in \|\mathrm{temp}_0(\mathsf{E})\|} h[\mathsf{E},0](u) \right) \cdot \left(\sum_{v \in \|\mathrm{temp}_m(\mathsf{E}')\|} h[\mathsf{E}',m](v) \right) = \\
&= \left(\sum_{u \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](u) \right) \cdot \left(\sum_{v \in \|\mathrm{temp}(\mathsf{E}')\|} h[\mathsf{E}'](v) \right) = \|\mathsf{E}\| \cdot \|\mathsf{E}'\|.
\end{aligned}$$

This also proves that the multiplication of rational expressions is associative in the sense that

$$\|(\mathsf{E}_1 \cdot (\mathsf{E}_2 \cdot \mathsf{E}_3))\| = \|((\mathsf{E}_1 \cdot \mathsf{E}_2) \cdot \mathsf{E}_3)\|$$

holds for each triple of rational expressions $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$ such that $\|\mathsf{E}_1\|$, $\|\mathsf{E}_2\|$, and $\|\mathsf{E}_3\|$ are not $\perp$. Thus, in case we are interested just in the element denoted by the expression, we may write $\mathsf{E}_1 \cdot \mathsf{E}_2 \cdot \mathsf{E}_3$ to denote *any* full parenthesisation of $\mathsf{E}_1 \cdot \mathsf{E}_2 \cdot \mathsf{E}_3$.

By induction, we shall extend this convention to products of more than three rational expressions, which also makes it possible to write E^n for the "n-the power of E", so that $\|\mathsf{E}^n\| = \|\mathsf{E}\|^n$.

For the star (E^*) of the expression E, we have

$$\|\mathrm{temp}((\mathsf{E}^*))\| = \|(\mathrm{temp}(\mathsf{E}) \cdot \mathsf{c}_m)^*\| = \|(\mathrm{temp}(\mathsf{E}) \cdot \mathsf{c}_m)\|^* = (\|\mathrm{temp}(\mathsf{E})\| \cdot c_m)^* = \bigcup_{i \in \mathbb{N}} (\|\mathrm{temp}(\mathsf{E})\| \cdot c_m)^i.$$

This union is disjoint, as $(\|\mathrm{temp}(\mathsf{E})\| \cdot \mathsf{c}_m)^i$ contains precisely i occurrences of c_m for each i in $\mathbb{N}$. Hence, if $\|(\mathsf{E}^*)\| \neq \perp$ – that is, if $(h[(\mathsf{E}^*)](w) \mid w \in \|\mathrm{temp}((\mathsf{E}^*))\|)$ is in $\mathcal{F}$ – and if we adopt the notational shortcuts

$$\sum_{x_1 \in X_1, \ldots, x_n \in X_n} \varphi(x_1, \ldots, x_n) := \sum_{(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n} \varphi(x_1, \ldots, x_n)$$

and

$$\sum_{x_1, \ldots, x_n \in X} \varphi(x_1, \ldots, x_n) := \sum_{(x_1, \ldots, x_n) \in X \times \ldots \times X} \varphi(x_1, \ldots, x_n),$$

then conditions (i) to (iv) of Definition 2.1 together with (4) give us

$$\begin{aligned}
\|(\mathsf{E}^*)\| &= \sum_{w \in \|\mathrm{temp}((\mathsf{E}^*))\|} h[(\mathsf{E}^*)](w) = \sum_{i \in \mathbb{N}} \sum_{w \in (\|\mathrm{temp}(\mathsf{E})\| \cdot c_m)^i} h[(\mathsf{E}^*)](w) = \\
&= \sum_{i \in \mathbb{N}} \sum_{w_1, \ldots, w_i \in \|\mathrm{temp}(\mathsf{E})\| \cdot c_m} h[(\mathsf{E}^*)](w_1) \cdot \ldots \cdot h[(\mathsf{E}^*)](w_i) = \\
&= \sum_{i \in \mathbb{N}} \sum_{x_1, \ldots, x_i \in \|\mathrm{temp}(\mathsf{E})\|} h[(\mathsf{E}^*)](x_1) \cdot h[(\mathsf{E}^*)](c_m) \cdot \ldots \cdot h[(\mathsf{E}^*)](x_i) \cdot h[(\mathsf{E}^*)](c_m) = \\
&= \sum_{i \in \mathbb{N}} \sum_{x_1, \ldots, x_i \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](x_1) \cdot h[\mathsf{E}](x_2) \cdot \ldots \cdot h[\mathsf{E}](x_i) = \\
&= \sum_{i \in \mathbb{N}} \sum_{\substack{x_1 \in \|\mathrm{temp}_0(\mathsf{E})\| \\ x_2 \in \|\mathrm{temp}_m(\mathsf{E})\| \\ \vdots \\ x_i \in \|\mathrm{temp}_{mi}(\mathsf{E})\|}} h[\mathsf{E},0](x_1) \cdot h[\mathsf{E},m](x_2) \cdot \ldots \cdot h[\mathsf{E},mi](x_i) = \\
&= \sum_{i \in \mathbb{N}} \sum_{x \in \|\mathrm{temp}(\mathsf{E}^i)\|} h[\mathsf{E}](x) = \sum_{i \in \mathbb{N}} \|\mathsf{E}^i\| = \sum_{i \in \mathbb{N}} \|\mathsf{E}\|^i,
\end{aligned}$$

all sums being over summable families, completing the proof. □

Remark 5.8. Note that $\|(\mathsf{E}^*)\|$ can be undefined even if $\|\mathsf{E}\|$ and $\|\mathsf{E}\|^*$ are defined. This is true for instance for $\mathsf{E} = (-1 + (1 + \mathsf{a}))$ interpreted over $\mathbb{Z}\langle\langle a^* \rangle\rangle$ with sums over locally finite families of series. The intuitive reason behind this is that semantics of rational expressions over summation semirings is defined by first forming a (possibly infinite) family of elements and then summing these elements up. It may happen that this infinite sum is undefined, while at the same time, it may be possible to sum up the family by parts – that is, to first partition the original family to subfamilies, next evaluate the sums of these subfamilies, and finally sum up the results for all subfamilies.

Corollary 5.9. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. Then the set of all elements of S denoted by rational S'-expressions over S is closed under sum and product.*

Proof. Follows immediately by Proposition 5.7. □

We shall say that a subset T of a summation semiring $(S, \mathcal{F}, \Phi)$ is *closed under well-defined star* if

$$a^* := \sum_{i \in \mathbb{N}} a^i$$

is in T whenever a is in T and $(a^i \mid i \in \mathbb{N})$ is in $\mathcal{F}$. We shall now prove one of the less satisfying properties of rational expressions over summation semirings: the set of semiring elements denoted by them might not be closed under well-defined star. Note that it is *not* possible to use the expression E of Remark 5.8 as a counterexample here, as clearly $\|\mathsf{E}\| = \|\mathsf{a}\|$. In fact, it is well known that the closure under well-defined star *does hold* in the setting of Remark 5.8. As a matter of fact, a counterexample needs to be constructed for relatively unusual S and S'.

Proposition 5.10. *There is a summation semiring $(S, \mathcal{F}, \Phi)$ and a subset S' of S containing 0 and 1 such that the set of all a in S denoted by rational S'-expressions over S is not closed under well-defined star.*

Proof. Let $\Sigma = \{a, b\}$ and $S = 2^{\Sigma^*}$. Let $\mathcal{F}$ consist of all families $(L_i \mid i \in I)$ of subsets of Σ^* such that $|L_i| = 1$ for finitely many i in I, and let

$$\Phi(L_i \mid i \in I) = \sum_{i \in I} L_i := \bigcup_{i \in I} L_i$$

for each family $(L_i \mid i \in I)$ in $\mathcal{F}$. It is easy to prove that $(S, \mathcal{F}, \Phi)$ is a summation semiring. Moreover, let $S' = 2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\}$ consist of all singleton subsets of Σ, identified with elements of Σ, together with the languages $\emptyset$ and $\{\varepsilon\}$. It is then straightforward to prove that the rational S'-expression $\mathsf{E} = (\mathsf{a} + \mathsf{b})$ over S denotes the language $\|\mathsf{E}\| = \{a, b\}$. The star of this language,

$$\|\mathsf{E}\|^* = \{a, b\}^* = \bigcup_{t \in \mathbb{N}} \{a, b\}^t$$

is clearly well-defined in $(S, \mathcal{F}, \Phi)$. However, there is no rational S'-expression E' over $(S, \mathcal{F}, \Phi)$ such that $\|\mathsf{E}'\| = \{a, b\}^*$. For if F was such expression, then it can be assumed to contain no atom $\emptyset$ – this can be proved by induction, utilising the fact that an infinite sum of empty sets is always an empty set in $(S, \mathcal{F}, \Phi)$. It follows that $\|\mathrm{temp}(\mathsf{F})\|$ would have to be infinite and

$$\|\mathsf{F}\| = \sum_{w \in \|\mathrm{temp}(\mathsf{F})\|} h[\mathsf{F}](w)$$

would be an infinite sum of singleton subsets of Σ^*, which is undefined in $(S, \mathcal{F}, \Phi)$. □

The following lemma states for rational expressions over summation semirings the same as Lemma 3.7 states for finite automata: if an element of a summation semiring S is a sum of homomorphic images of words in a rational language – not necessarily a language denoted by a template expression – then it is also denoted by some rational expression over S.

Lemma 5.11. *Let Σ be an alphabet, $L \subseteq \Sigma^*$ a rational language, $(S, \mathcal{F}, \Phi)$ a summation semiring, and $h\colon \Sigma^* \to (S, \cdot)$ a monoid homomorphism. If $(h(w) \mid w \in L)$ is in $\mathcal{F}$, then the semiring element*

$$\sum_{w \in L} h(w)$$

equals $\|\mathsf{E}\|$ for some rational S'-expression E over S with $S' = h(\Sigma) \cup \{0, 1\}$.

Proof. As L is a rational language, there is an *unambiguous* rational $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-expression F over 2^{Σ^*} such that $\|\mathsf{F}\| = L$ holds [22]. The statement of the lemma is trivially true for $L = \emptyset$; we may thus assume that $L \neq \emptyset$ and that $h[\mathsf{F}](c)$ is in $2_1^\Sigma \cup \{\varepsilon\}$ for each c in $\Sigma[\mathsf{F}]$. Under this assumption, let $f\colon \Sigma[\mathsf{F}]^* \to \Sigma^*$ be a homomorphism defined for all c in $\Sigma[\mathsf{F}]$ by $f(c) = z$, where z in $\Sigma \cup \{\varepsilon\}$ satisfies $h[\mathsf{F}](c) = \{z\}$, so that

$$L = \bigcup_{x \in \|\mathrm{temp}(\mathsf{F})\|} \{f(x)\};$$

this union is disjoint by unambiguity of F. It is clear that there is a unique rational S'-expression E over S such that $\Sigma[\mathsf{E}] = \Sigma[\mathsf{F}]$, $\mathrm{temp}(\mathsf{E}) = \mathrm{temp}(\mathsf{F})$, and $h[\mathsf{E}]\colon \Sigma[\mathsf{E}]^* \to (S, \cdot)$ is given by $h[\mathsf{E}] = h \circ f$. It then follows by conditions (ii) and (i) of Definition 2.1 that the family $(h[\mathsf{E}](x) \mid x \in \|\mathrm{temp}(\mathsf{E})\|)$ is summable and that

$$\|\mathsf{E}\| = \sum_{x \in \|\mathrm{temp}(\mathsf{E})\|} h[\mathsf{E}](x) = \sum_{x \in \|\mathrm{temp}(\mathsf{F})\|} h(f(x)) = \sum_{w \in L} h(w).$$

This completes the proof. □

We are now ready to prove the main result of this section: the equivalence of rational expressions and finite automata over summation semirings.

Theorem 5.12. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing both 0 and 1, and a an element of S. The following statements are equivalent:*

(i) There is a finite S'-automaton $\mathcal{A}$ over S such that $\|\mathcal{A}\| = a$.

(ii) There is a rational S'-expression E over S such that $\|\mathsf{E}\| = a$.

(iii) There is an alphabet Σ, a rational language $L \subseteq \Sigma^*$, and a monoid homomorphism $h\colon \Sigma^* \to (S, \cdot)$ such that $h(\Sigma) \subseteq S'$, $(h(w) \mid w \in L)$ is in $\mathcal{F}$, and

$$a = \sum_{w \in L} h(w).$$

Proof. By Lemma 3.7 and Lemma 5.11, both (i) and (ii) follow from (iii). The converse implications follow directly by the fact that $\|\mathrm{temp}(\mathcal{A})\|$ and $\|\mathrm{temp}(\mathsf{E})\|$ are rational languages for each finite S'-automaton $\mathcal{A}$ and each rational S'-expression E over S. □

We shall say that an element of a summation semiring S is an *S'-rational element* if it is denoted by some rational S'-expression over S. By Theorem 5.12, this is equivalent to the element being a behaviour of some finite S'-automaton over S.

Definition 5.13. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and a in S. We shall say that a is an *S'-rational element of S* if there is a rational S'-expression E over S with $\|\mathsf{E}\| = a$.

Notation 5.14. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. Then we shall denote by $\mathrm{Rat}(S', S)$ the set of all S'-rational elements of S.

6. Right-Linear Systems over Summation Semirings

We shall now define right-linear systems over summation semirings and establish their equivalence with finite automata and rational expressions. These will extend, e.g., the proper linear systems of [24]. Moreover, they can also be seen as "right-linear grammars over semirings"; the definition of their canonical solutions is largely inspired by this point of view.

Definition 6.1. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. A *right-linear S'-system* over S is a quadruple $(n, \mathbf{y}, A, \mathbf{b})$, where n is a positive integer, $\mathbf{y} = (y_1, \ldots, y_n)^T$ is a vector of variables, $A = (a_{i,j})_{n \times n}$ is an $n \times n$ matrix such that $a_{i,j}$ is in S' for $i, j = 1, \ldots, n$, and $\mathbf{b} = (b_1, \ldots, b_n)^T$ is a vector such that b_i is in S' for $i = 1, \ldots, n$. We shall usually write the right-linear system $(n, \mathbf{y}, A, \mathbf{b})$ as

$$\mathbf{y} = A\mathbf{y} + \mathbf{b}. \tag{5}$$

Moreover, we shall say that a vector $\mathbf{x}$ in S^n is a *solution* to the system (5) if $\mathbf{x} = A\mathbf{x} + \mathbf{b}$ holds.

We shall now define the *canonical solution* of a right-linear system via a template system over a suitable semiring of formal languages. This definition parallels the definition of behaviour of finite automata. Recall that if X is a set, then 2_1^X denotes the set of all singleton subsets of X and the element $\{x\}$ of 2_1^X can be identified with x in X.

Definition 6.2. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ a right-linear S'-system over S. Let $\Sigma = \{c_{i,j} \mid i, j \in \mathbb{N};\ 1 \le i, j \le n\} \cup \{d_i \mid i \in \mathbb{N};\ 1 \le i \le n\}$. A *template system* corresponding to $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ is a right-linear $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-system

$$\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})$$

over 2^{Σ^*}, where

$$\mathrm{temp}(A) = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} \qquad \text{and} \qquad \mathrm{temp}(\mathbf{b}) = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Let $(\|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_1, \ldots, \|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_n)^T$ be the least solution to the template system $\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})$, guaranteed to exist by the Ginsburg-Rice theorem for context-free languages [13]; see also [20]. Moreover, for $A = (a_{i,j})_{n \times n}$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$, let

$$h[A, \mathbf{b}] \colon \Sigma^* \to (S, \cdot)$$

be a monoid homomorphism such that $h[A, \mathbf{b}](c_{i,j}) = a_{i,j}$ for $i, j = 1, \ldots, n$ and $h[A, \mathbf{b}](d_i) = b_i$ for $i = 1, \ldots, n$. Then $\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i$ is given, for $i = 1, \ldots, n$, by

$$\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i := \sum_{w \in \|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_i} h[A, \mathbf{b}](w)$$

if $(h[A, \mathbf{b}](w) \mid w \in \|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_i)$ is in $\mathcal{F}$ and by $\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i = \bot$ otherwise. We shall call the vector

$$(\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1, \ldots, \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_n)^T$$

the *canonical solution* to $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ and say that it is *completely defined* if $\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i \neq \bot$ holds for $i = 1, \ldots, n$.

Remark 6.3. Similarly as for finite automata and for rational expressions, the notation $\|\cdot\|$ is used in Definition 6.2 with two different meanings: it can denote both the canonical solution to a right-linear system over a summation semiring, or the least solution to a template system. As each template system is at the same time a system over a summation semiring, this presents a potential conflict in notation. However, it is easy to see that the canonical solution to each template system is precisely its least solution; the use of the same notation in both cases is thus justified.

Remark 6.4. Note that right-linear grammars corresponding to template systems are always unambiguous, as each right-hand side of a production rule in such grammar starts with a distinct terminal symbol.

Example 6.5. Consider the following right-linear system over the summation semiring of formal power series $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ for $\Sigma = \{a, b\}$, with sums over locally finite families of series:

$$\mathbf{y} = \begin{pmatrix} 2a & 1 \\ 0 & 2b \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6}$$

The corresponding template system is then given by

$$\mathbf{y} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} \mathbf{y} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

and it is obvious that its least solution is given by $(L_1, L_2)^T$, where

$$L_1 = (\{c_{1,1}\} \cup c_{1,2}c_{2,2}^*c_{2,1})^*(\{d_1\} \cup c_{1,2}c_{2,2}^*d_2),$$
$$L_2 = (\{c_{2,2}\} \cup c_{2,1}c_{1,1}^*c_{1,2})^*(\{d_2\} \cup c_{2,1}c_{1,1}^*d_1).$$

Moreover, if we denote the matrix of the system (6) by A and the vector $(0, 1)^T$ by $\mathbf{b}$, the homomorphism $h[A, \mathbf{b}]$ is given as follows:

$$h[A,\mathbf{b}](c_{1,1}) = 2a, \qquad h[A,\mathbf{b}](c_{1,2}) = 1, \qquad h[A,\mathbf{b}](d_1) = 0,$$
$$h[A,\mathbf{b}](c_{2,1}) = 0, \qquad h[A,\mathbf{b}](c_{2,2}) = 2b, \qquad h[A,\mathbf{b}](d_2) = 1.$$

Let $\Gamma = \{c_{1,1}, c_{1,2}, c_{2,2}, d_2\}$ be the alphabet of symbols, for which the homomorphism $h[A, \mathbf{b}]$ evaluates to a non-zero value. Then the canonical solution of (6) is given by $(r_1, r_2)^T$, where

$$r_1 = \sum_{w \in L_1} h[A,\mathbf{b}](w) = \sum_{w \in L_1 \cap \Gamma^*} h[A,\mathbf{b}](w) + \sum_{w \in L_1 - \Gamma^*} h[A,\mathbf{b}](w) = \left(\sum_{w \in c_{1,1}^* c_{1,2} c_{2,2}^* d_2} h[A,\mathbf{b}](w) \right) + 0 =$$
$$= \sum_{i,j \in \mathbb{N}} h[A,\mathbf{b}](c_{1,1}^i c_{1,2} c_{2,2}^j d_2) = \sum_{i,j \in \mathbb{N}} 2^{i+j} a^i b^j,$$

$$r_2 = \sum_{w \in L_2} h[A,\mathbf{b}](w) = \sum_{w \in L_2 \cap \Gamma^*} h[A,\mathbf{b}](w) + \sum_{w \in L_2 - \Gamma^*} h[A,\mathbf{b}](w) = \left(\sum_{w \in c_{2,2}^* d_2} h[A,\mathbf{b}](w) \right) + 0 =$$
$$= \sum_{j \in \mathbb{N}} h[A,\mathbf{b}](c_{2,2}^j d_2) = \sum_{j \in \mathbb{N}} 2^j b^j.$$

Let us now prove that each *completely defined* canonical solution to a right-linear system is also a solution in the sense of Definition 6.1.

Proposition 6.6. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing $0, 1$, and $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ a right-linear S'-system over S such that its canonical solution $\mathbf{s} = (\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1, \ldots, \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_n)^T$ is completely defined. Then $\mathbf{s}$ is a solution to $\mathbf{y} = A\mathbf{y} + \mathbf{b}$, i.e., $\mathbf{s} = A\mathbf{s} + \mathbf{b}$ holds.*

Proof. It clearly suffices to prove that

$$\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i = b_i + \sum_{k=1}^{n} a_{i,k} \cdot \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_k \tag{7}$$

holds for $i = 1, \ldots, n$. Let i be fixed. Then

$$\|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_i = \{d_i\} \cup \bigcup_{k=1}^{n} c_{i,k} \cdot \|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_k$$

with the unions being disjoint. Thus, it follows by conditions (*i*) to (*iv*) of Definition 2.1 that

$$\begin{aligned}\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_i &= \sum_{w \in \|\mathbf{y}=\mathrm{temp}(A)\mathbf{y}+\mathrm{temp}(\mathbf{b})\|_i} h[A,\mathbf{b}](w) = \\ &= h[A,\mathbf{b}](d_i) + \sum_{k=1}^{n} h[A,\mathbf{b}](c_{i,k}) \cdot \sum_{w \in \|\mathbf{y}=\mathrm{temp}(A)\mathbf{y}+\mathrm{temp}(\mathbf{b})\|_k} h[A,\mathbf{b}](w) = \\ &= b_i + \sum_{k=1}^{n} a_{i,k} \cdot \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_k,\end{aligned}$$

proving (7) and hence also the proposition. □

Let us now prove for right-linear systems a result analogous to Lemma 3.7 for finite automata and to Lemma 5.11 for rational expressions: if an element of a summation semiring can be expressed as a sum of homomorphic images of words from a rational language, then it is a first component of a canonical solution to some right-linear system (this canonical solution might not be completely defined).

Lemma 6.7. *Let* Σ *be an alphabet,* $L \subseteq \Sigma^*$ *a rational language,* $(S, \mathcal{F}, \Phi)$ *a summation semiring, and* $h\colon \Sigma^* \to (S, \cdot)$ *a monoid homomorphism. If* $(h(w) \mid w \in L)$ *is in* $\mathcal{F}$*, then the semiring element*

$$\sum_{w \in L} h(w)$$

equals $\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1$ *for some right-linear* S'*-system* $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ *over* S *with* $S' = h(\Sigma) \cup \{0, 1\}$.

Proof. The language L is rational; it is thus generated by some unambiguous right-linear grammar [15]. Assume without loss of generality that the rules of this grammar are all either of the form $\xi \to z\nu$ for some nonterminals ξ, ν and some z in $\Sigma \cup \{\varepsilon\}$, where each such z is *uniquely* determined by ξ and ν (this can be assured by maintaining a copy of each nonterminal for each z in $\Sigma \cup \{\varepsilon\}$), or of the form $\xi \to z$ for some nonterminal ξ and some z in $\Sigma \cup \{\varepsilon\}$, where each such z is *uniquely* determined by ξ (this can be assured by introducing a new nonterminal for each z in $\Sigma \cup \{\varepsilon\}$ and possibly some chain rules). By virtue of the classical Ginsburg-Rice theorem [13], this right-linear grammar corresponds to some right-linear $(2^{\Sigma}_1 \cup \{\emptyset, \{\varepsilon\}\})$-system $\mathbf{y} = A'\mathbf{y} + \mathbf{b}'$ over 2^{Σ^*} such that $\|\mathbf{y} = A'\mathbf{y} + \mathbf{b}'\|_1 = L$. Let $A' = (\alpha_{i,j})_{n \times n}$ and $\mathbf{b}' = (\beta_1, \ldots, \beta_n)^T$ for some positive integer n.

Let us now consider a right-linear S'-system $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ over S with $A = (a_{i,j})_{n \times n}$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$ defined by

$$a_{i,j} = \begin{cases} h(z) & \text{if } \alpha_{i,j} = \{z\} \text{ for some } z \in \Sigma \cup \{\varepsilon\} \\ 0 & \text{if } \alpha_{i,j} = \emptyset \end{cases}$$

for $i, j = 1, \ldots, n$ and by

$$b_i = \begin{cases} h(z) & \text{if } \beta_i = \{z\} \text{ for some } z \in \Sigma \cup \{\varepsilon\} \\ 0 & \text{if } \beta_i = \emptyset \end{cases}$$

for $i = 1, \ldots, n$. Clearly, $\|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_1$ is the same as $\|\mathbf{y} = \mathrm{temp}(A')\mathbf{y} + \mathrm{temp}(\mathbf{b}')\|_1$ – let us denote this language by T. Here, $\mathrm{temp}(A) = (c_{i,j})_{n \times n}$ and $\mathrm{temp}(\mathbf{b}) = (d_1, \ldots, d_n)^T$. Let us denote by Γ the alphabet $\Gamma = \{c_{i,j} \mid i, j \in \mathbb{N};\ 1 \le i, j \le n\} \cup \{d_i \mid i \in \mathbb{N};\ 1 \le i \le n\}$. Moreover, let us denote by $\chi(T)$ the language

$$\chi(T) = \{w \in T \mid h[A', \mathbf{b}'](w) \neq \emptyset\}$$

and by $\chi(\Gamma)$ the alphabet $\chi(\Gamma) = \{c \in \Gamma \mid h[A', \mathbf{b}'](c) \neq \emptyset\}$. Clearly, $\chi(T)$ is a language over $\chi(\Gamma)$. Let $f\colon \chi(\Gamma)^* \to \Sigma^*$ be a homomorphism defined for all c in $\chi(\Gamma)$ by $f(c) = z$ if $h[A', \mathbf{b}'](c) = \{z\}$, so that $h(f(x)) = h[A, \mathbf{b}](x)$ for all x in $\chi(\Gamma)^*$ and

$$L = \bigcup_{x \in \chi(T)} \{f(x)\}$$

with the union being disjoint by unambiguity of the grammar for L. It then follows by conditions (i) to (iv) of Definition 2.1 that $(h[A, \mathbf{b}](x) \mid x \in T)$ is in $\mathcal{F}$ and

$$\|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1 = \sum_{x \in T} h[A, \mathbf{b}](x) = \sum_{x \in \chi(T)} h[A, \mathbf{b}](x) + \sum_{x \in T - \chi(T)} 0 = \sum_{x \in \chi(T)} h[A, \mathbf{b}](x) =$$
$$= \sum_{x \in \chi(T)} h(f(x)) = \sum_{w \in L} h(w).$$

This completes the proof. □

We are now ready to state the main result of this section: the equivalence of right-linear systems with finite automata and rational expressions.

Theorem 6.8. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing both 0 and 1, and a an element of S. Then a is in $\mathrm{Rat}(S', S)$ if and only if $a = \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1$ for some right-linear S'-system $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ over S.*

Proof. By Theorem 5.12, it is enough to prove that $a = \|\mathbf{y} = A\mathbf{y} + \mathbf{b}\|_1$ for some right-linear S'-system $\mathbf{y} = A\mathbf{y} + \mathbf{b}$ over S if and only if

$$a = \sum_{w \in L} h(w)$$

for some rational language $L \subseteq \Sigma^*$ and a monoid homomorphism $h\colon \Sigma^* \to (S, \cdot)$ such that $h(\Sigma) \subseteq S'$ and $(h(w) \mid w \in L)$ is in $\mathcal{F}$. However, the "if part" of this statement follows by Lemma 6.7, while for the "only if part" it suffices to take $L = \|\mathbf{y} = \mathrm{temp}(A)\mathbf{y} + \mathrm{temp}(\mathbf{b})\|_1$ and $h = h[A, \mathbf{b}]$. □

7. MSO Logics over Summation Semirings

The behaviour of a finite automaton $\mathcal{A}$ over a summation semiring has been defined by first associating a template automaton $\mathrm{temp}(\mathcal{A})$ over some suitable alphabet to $\mathcal{A}$ and next taking, if it exists, the sum of homomorphic images of words from the language recognised by $\mathrm{temp}(\mathcal{A})$. A similar approach has been used in the definitions of semantics for rational expressions and of canonical solutions to right-linear systems.

We shall now use this method of "homomorphic transition" from template languages to summation semiring elements to generalise the classical monadic second-order (MSO) logics on words of Büchi and Elgot [3, 10, 9, 16, 19, 25] to *monadic second-order logics over summation semirings* that are expressive equivalents of finite automata, rational expressions, and right-linear systems.

The concept of MSO logics over summation semirings, as introduced in what follows, should be distinguished from the concept of weighted MSO logics of Droste and Gastin [4, 5]. When restricted to semirings of formal power series, our MSO logics do *not* generalise the weighted MSO logics. In fact, already the philosophy behind both concepts is different: while weighted MSO logics of Droste and Gastin [4, 5] extend the classical MSO logics on words by a possibility of "emitting weights", MSO logics over summation semirings are no longer logics on words, but "logics on factorisations of semiring elements". Hence, while weighted MSO logics are suitable for modelling quantitative properties related to definability via MSO logics on words such as the "number of proofs" that a word belongs to a language, MSO logics over a summation semiring S can be used to specify sets of "valid annotated products" $a_1^{[i_1]} \cdot a_2^{[i_2]} \cdot \ldots \cdot a_n^{[i_n]}$, where $a_1, \ldots, a_n$ are elements of some subset S' of S and $i_1, \ldots, i_n$ are "annotations" from some finite subset of $\mathbb{N}$. Different "annotations" of elements of S' correspond to different predicates used to specify them – MSO logics over summation semirings thus have a predicate for each element of S' and each "annotation" in $\mathbb{N}$. The semantics of an MSO sentence φ is then defined to be, if it exists, the sum of $a_1 \cdot a_2 \cdot \ldots \cdot a_n$ over all "valid annotated products" $a_1^{[i_1]} \cdot a_2^{[i_2]} \cdot \ldots \cdot a_n^{[i_n]}$ specified by φ. Furthermore, we shall see that MSO logics over summation semirings of power series are not even expressive equivalents of weighted MSO logics: while MSO-definable elements of summation semirings coincide with the rational elements, this does not hold without further restrictions for MSO-definability via weighted logics, which can also define power series that are not rational.

Definition 7.1. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing both 0 and 1. The *language* of the *monadic second-order logic over S' and S* (abbreviated $\mathrm{MSO}(S', S)$) is built upon an infinite alphabet consisting of:

a) Symbols $x, y, z, \ldots$, possibly with indices or other "decorations", for first-order variables.

b) Symbols $X, Y, Z, \ldots$, possibly with indices or other "decorations", for set variables.

c) Symbols $\neg$ and $\vee$ for logical connectives (negation and disjunction).

d) A symbol $\exists$ for the existential quantifier.

e) A symbol $\leq$ denoting the order relation on indices.

f) For each a in S' and each i in $\mathbb{N}$, a symbol for a unary predicate $P_a^{[i]}$.

g) A symbol $\in$ for set membership.

h) Symbols (and) for parentheses.

The set of all *well-formed formulae* in $\mathrm{MSO}(S', S)$ is defined as follows:

1. If x, y are first-order variables, X a set variable, a an element of S', and i in $\mathbb{N}$, then $x \leq y$, $x \in X$, and $P_a^{[i]}(x)$ are well-formed formulae in $\mathrm{MSO}(S', S)$ called *atomic formulae*.
2. If φ, ψ are well-formed formulae in $\mathrm{MSO}(S', S)$, x a first-order variable, and X a set variable, then $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\exists x. \varphi)$, and $(\exists X. \varphi)$ are well-formed formulae in $\mathrm{MSO}(S', S)$.
3. Nothing else is a well-formed formula in $\mathrm{MSO}(S', S)$.

We shall now view $S' \times \mathbb{N}$ – that is, the set of annotated elements of S' – as an infinite alphabet Σ for a while, and define the semantics of a well-formed formula in $\mathrm{MSO}(S', S)$ via the notion of *satisfaction* of a formula on a word w in Σ^*, using the standard semantics for MSO logic on words. We shall thus in fact interpret the formulae of $\mathrm{MSO}(S', S)$ in the relational structures $\underline{w} = (\{1, \ldots, |w|\}, \preceq^{(w)}, (\mathcal{P}_{(a,i)}^{(w)})_{(a,i)\in\Sigma})$ for w in Σ^*, where $\preceq^{(w)}$ is the usual order on $\{1, \ldots, |w|\}$ and

$$\mathcal{P}_{(a,i)}^{(w)}(k) = \begin{cases} 1 & \text{if } (a,i) \text{ is the } k\text{-th symbol of } w \\ 0 & \text{otherwise} \end{cases}$$

for each (a, i) in Σ and each k in $\{1, \ldots, |w|\}$.

Definition 7.2. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and Σ an infinite alphabet $\Sigma = S' \times \mathbb{N}$. For all well-formed formulae φ in $\mathrm{MSO}(S', S)$ let the *template formula* $\mathrm{temp}(\varphi)$ in $\mathrm{MSO}(\Sigma_\varphi)$, for some finite $\Sigma_\varphi \subseteq \Sigma$, be defined inductively as follows:

1. If x, y are first-order variables, X a set variable, a an element of S', and i in $\mathbb{N}$, then $\mathrm{temp}(x \leq y)$ is $x \leq y$, $\mathrm{temp}(x \in X)$ is $x \in X$, and $\mathrm{temp}(P_a^{[i]}(x))$ is $P_{(a,i)}(x)$.
2. If φ, ψ are well-formed formulae in $\mathrm{MSO}(S', S)$, x a first-order variable, and X a set variable, then $\mathrm{temp}((\neg\varphi))$ is $(\neg\mathrm{temp}(\varphi))$, $\mathrm{temp}((\varphi \vee \psi))$ is $(\mathrm{temp}(\varphi) \vee \mathrm{temp}(\psi))$, $\mathrm{temp}((\exists x. \varphi))$ is $(\exists x. \mathrm{temp}(\varphi))$, and $\mathrm{temp}((\exists X. \varphi))$ is $(\exists X. \mathrm{temp}(\varphi))$.

We shall say that φ is a *sentence* in $\mathrm{MSO}(S', S)$ if $\mathrm{temp}(\varphi)$ is a sentence in $\mathrm{MSO}(\Sigma_\varphi)$, i.e., if $\mathrm{temp}(\varphi)$ contains no free variables. For each w in Σ^* and each sentence φ in $\mathrm{MSO}(S', S)$, we shall write

$$\underline{w} \models \mathrm{temp}(\varphi)$$

if the sentence $\mathrm{temp}(\varphi)$ is, under the standard notion of satisfiability for MSO formulae on words [25], satisfied by $\underline{w}$ (to make the semantics precise, we shall assume that Σ_φ is the *smallest* alphabet such that temp(φ) is a formula in MSO(Σ_φ), and allow $\underline{w} \models$ temp(φ) only if w is in Σ_φ^*). Moreover, let $\|\mathrm{temp}(\varphi)\|$ be the set

$$\|\mathrm{temp}(\varphi)\| = \{w \in \Sigma^* \mid \underline{w} \models \mathrm{temp}(\varphi)\}$$

and $h[\varphi]\colon \Sigma^* \to (S, \cdot)$ a monoid homomorphism given for each (a, i) in Σ by $h[\varphi]((a, i)) = a$. The *element of S defined by φ* is then given by

$$\|\varphi\| := \sum_{w \in \|\mathrm{temp}(\varphi)\|} h[\varphi](w)$$

in case $(h[\varphi](w) \mid w \in \|\mathrm{temp}(\varphi)\|)$ is in $\mathcal{F}$; we shall write $\|\varphi\| = \bot$ otherwise.

Remark 7.3. Similarly as for finite automata, rational expressions, and right-linear systems, we use $\|\cdot\|$ to denote both the element defined by an MSO formula over a summation semiring and the language defined by a template formula. However, it is easy to see that this notation is consistent – $\|\mathrm{temp}(\varphi)\|$ is the same language regardless if temp(φ) is viewed as an MSO formula on words or an MSO formula over a summation semiring of formal languages.

In the following examples, we also use logical connectives $\wedge$ and $\to$ in addition to $\neg$ and $\vee$. In particular, for each pair of formulae φ, ψ, we interpret $(\varphi \wedge \psi)$ as a shortcut for the formula $(\neg((\neg\varphi) \vee (\neg\psi)))$ and $(\varphi \to \psi)$ as a shortcut for $((\neg\varphi) \vee \psi)$.

Example 7.4. Consider the complete semiring $S = (2^{\mathbb{R}}, \cup, \cdot, \emptyset, \{1\})$, i.e., the semiring of subsets of the monoid $(\mathbb{R}, \cdot, 1)$. Let S' be a subset of S containing $\emptyset$, $\{1\}$, $\{2\}$, and $\{1/2\}$; from now on, we shall identify singleton subsets $\{x\}$ of $\mathbb{R}$ with their single element x.

Let us take the following formula in MSO(S', S) (with some parentheses omitted for clarity):

$$\varphi = \forall x. \, (P_2^{[1]}(x) \vee P_{1/2}^{[1]}(x));$$

this is a first-order formula, in fact. The template formula for φ is then given by

$$\mathrm{temp}(\varphi) = \forall x. \, (P_{(2,1)}(x) \vee P_{(1/2,1)}(x)),$$

and it can be readily observed that this formula defines the language

$$\|\mathrm{temp}(\varphi)\| = \{(2, 1), (1/2, 1)\}^*.$$

Given $h[\varphi]((2, 1)) = 2$ and $h[\varphi]((1/2, 1)) = 1/2$, we see that

$$\|\varphi\| = \sum_{w \in \|\mathrm{temp}(\varphi)\|} h[\varphi](w) = \bigcup_{w \in \{(2,1),(1/2,1)\}^*} h[\varphi](w) = \{2^n \mid n \in \mathbb{Z}\}.$$

Example 7.5. Let us consider the semiring $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ with $\Sigma = \{a, b\}$ and sums over locally finite families of series, and let us construct an MSO formula for the series

$$r = 4a + \sum_{n \in \mathbb{N}} 2^n (ab)^n.$$

In the same way as in the usual MSO logic on words, we can define a formula FIRST(x) for any first-order variable x, which is satisfied whenever x takes the value 1. In particular, we may write

$$\mathrm{FIRST}(x) := \forall y. \, (x \leq y).$$

Similarly, we may define a formula LAST(x), which is satisfied on a word w whenever x takes the value $|w|$, and a formula SUCC(x, y), which is satisfied whenever $y = x + 1$.

The series r is then defined by the (first-order) formula

$$\begin{aligned}\varphi = (\exists x.\,(\mathrm{FIRST}(x) \wedge \mathrm{LAST}(x) \wedge P_{4a}^{[1]}(x))) \vee \\ \vee (\forall y.\,((P_{2a}^{[1]}(y) \vee P_{b}^{[1]}(y)) \wedge (\mathrm{FIRST}(y) \to P_{2a}^{[1]}(y)) \wedge (\mathrm{LAST}(y) \to P_{b}^{[1]}(y)) \wedge \\ \wedge \forall z.\,(\mathrm{SUCC}(y,z) \to ((P_{2a}^{[1]}(y) \wedge P_{b}^{[1]}(z)) \vee (P_{b}^{[1]}(y) \wedge P_{2a}^{[1]}(z)))))).\end{aligned}$$

Indeed, the template formula for φ is given by

$$\begin{aligned}\mathrm{temp}(\varphi) = (\exists x.\,(\mathrm{FIRST}(x) \wedge \mathrm{LAST}(x) \wedge P_{(4a,1)}(x))) \vee \\ \vee (\forall y.\,((P_{(2a,1)}(y) \vee P_{(b,1)}(y)) \wedge (\mathrm{FIRST}(y) \to P_{(2a,1)}(y)) \wedge (\mathrm{LAST}(y) \to P_{(b,1)}(y)) \wedge \\ \wedge \forall z.\,(\mathrm{SUCC}(y,z) \to ((P_{(2a,1)}(y) \wedge P_{(b,1)}(z)) \vee (P_{(b,1)}(y) \wedge P_{(2a,1)}(z)))))).\end{aligned}$$

The formula $\mathrm{temp}(\varphi)$ defines the language

$$\|\mathrm{temp}(\varphi)\| = \{(4a,1)\} \cup ((2a,1)(b,1))^*.$$

Moreover, $h[\varphi]((4a,1)) = 4a$, $h[\varphi]((2a,1)) = 2a$, and $h[\varphi]((b,1)) = b$. As a result, we obtain

$$\|\varphi\| = \sum_{w \in \|\mathrm{temp}(\varphi)\|} h[\varphi](w) = h[\varphi]((4a,1)) + \sum_{n\in\mathbb{N}} h[\varphi](((2a,1)(b,1))^n) = 4a + \sum_{n\in\mathbb{N}} 2^n(ab)^n = r.$$

Example 7.6. Annotations can be thought of as a "device for resolving ambiguity" of formulae. Consider for instance

$$\psi_1 = P_a^{[1]}(x) \vee P_a^{[1]}(x) \qquad \text{and} \qquad \psi_2 = P_a^{[1]}(x) \vee P_a^{[2]}(x)$$

over the semiring $\mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ with $\Sigma = \{a,b\}$ and sums over locally finite families of series (some parentheses have been omitted for clarity). The formula ψ_1 is equivalent to $P_a^{[1]}(x)$. However, in the presence of multiplicities from $\mathbb{N}$, it is also natural to use disjunction as addition in the underlying semiring, which is what the second formula ψ_2 does.

These formulae can for instance be used in sentences

$$\begin{aligned}\varphi_1 &= (\exists x.\,\mathrm{FIRST}(x)) \wedge (\forall x.\,(\mathrm{FIRST}(x) \wedge \psi_1)),\\ \varphi_2 &= (\exists x.\,\mathrm{FIRST}(x)) \wedge (\forall x.\,(\mathrm{FIRST}(x) \wedge \psi_2)),\end{aligned}$$

where $\mathrm{FIRST}(x)$ is defined in the same way as in Example 7.5. The template expressions for φ_1, φ_2 are then given by

$$\begin{aligned}\mathrm{temp}(\varphi_1) &= (\exists x.\,\mathrm{FIRST}(x)) \wedge (\forall x.\,(\mathrm{FIRST}(x) \wedge (P_{(a,1)}(x) \vee P_{(a,1)}(x)))),\\ \mathrm{temp}(\varphi_2) &= (\exists x.\,\mathrm{FIRST}(x)) \wedge (\forall x.\,(\mathrm{FIRST}(x) \wedge (P_{(a,1)}(x) \vee P_{(a,2)}(x)))),\end{aligned}$$

hence

$$\|\mathrm{temp}(\varphi_1)\| = \{(a,1)\} \qquad \text{and} \qquad \|\mathrm{temp}(\varphi_2)\| = \{(a,1),(a,2)\}.$$

As a result, given $h[\varphi_1]((a,1)) = h[\varphi_2]((a,1)) = h[\varphi_2]((a,2)) = a$, we obtain

$$\|\varphi_1\| = a \qquad \text{and} \qquad \|\varphi_2\| = a + a = 2a.$$

Let us now prove for MSO formulae over summation semirings a lemma analogous to Lemma 3.7 for finite automata, Lemma 5.11 for rational expressions, and Lemma 6.7 for right-linear systems: if an element of a summation semiring can be expressed as a sum of homomorphic images of words from a rational language, then it is MSO-definable.

Lemma 7.7. *Let Σ be an alphabet, $L \subseteq \Sigma^*$ a rational language, $(S, \mathcal{F}, \Phi)$ a summation semiring, and $h\colon \Sigma^* \to (S, \cdot)$ a monoid homomorphism. If $(h(w) \mid w \in L)$ is in $\mathcal{F}$, then the semiring element*

$$\sum_{w \in L} h(w)$$

equals $\|\varphi\|$ for some monadic second-order sentence φ over S' and S, where $S' = h(\Sigma) \cup \{0, 1\}$.

Proof. Suppose that $\Sigma = \{c_1, \ldots, c_n\}$ for some n in $\mathbb{N}$. Let $\Gamma = S' \times \{1, \ldots, n\}$ and $f : \Sigma^* \to \Gamma^*$ a homomorphism defined for all i in $\{1, \ldots, n\}$ by $f(c_i) = (h(c_i), i)$; f is clearly injective. As L is rational and the class of rational languages is closed under homomorphism, there is an MSO(Γ)-sentence ψ such that $\|\psi\| = f(L)$. Then there is a unique MSO(S', S)-sentence φ such that $\mathrm{temp}(\varphi) = \psi$ and $h[\varphi]\colon (S' \times \mathbb{N})^* \to (S, \cdot)$ is defined for all (a, i) in $S' \times \mathbb{N}$ by $h[\varphi]((a, i)) = a$. Clearly $h[\varphi](f(w)) = h(w)$ for all w in Σ^*. As a result, it follows by conditions (ii) and (i) of Definition 2.1 and by injectivity of f that

$$\|\varphi\| = \sum_{x \in \|\mathrm{temp}(\varphi)\|} h[\varphi](x) = \sum_{x \in \|\psi\|} h[\varphi](x) = \sum_{x \in f(L)} h[\varphi](x) = \sum_{w \in L} h[\varphi](f(w)) = \sum_{w \in L} h(w),$$

all sums being over summable families. The lemma is proved. □

We may now finally prove the equivalence of MSO-definability and rationality for elements of summation semirings.

Theorem 7.8. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing both 0 and 1, and a an element of S. Then a is in $\mathrm{Rat}(S', S)$ if and only if $a = \|\varphi\|$ holds for some monadic second-order sentence φ over S' and S.*

Proof. By Theorem 5.12, it is suffices to prove that $a = \|\varphi\|$ for some sentence φ in MSO(S', S) if and only if

$$a = \sum_{w \in L} h(w)$$

for some rational language $L \subseteq \Sigma^*$ and a monoid homomorphism $h\colon \Sigma^* \to (S, \cdot)$ such that $h(\Sigma) \subseteq S'$ and $(h(w) \mid w \in L)$ is in $\mathcal{F}$. However, the "if part" of this statement follows by Lemma 7.7 and the "only if part" is proved by taking $L = \|\mathrm{temp}(\varphi)\|$ and $h = h[\varphi]$. □

8. Partial Evaluation Semantics for Finite Automata

Our definition of behaviour of finite automata over summation semirings is a formalisation of the following idea: the semiring element realised by an automaton is a (possibly) infinite sum of elements corresponding to valid runs of the automaton, where the element corresponding to each run is given by a finite product of semiring elements. This approach to defining elements realised by automata over semirings is possible and well known both for automata over complete semirings and for proper weighted automata over arbitrary coefficient semirings [7]. Finite automata over summation semirings thus generalise both.

However, as already anticipated in Section 3, there are classes of finite automata over summation semirings, for which the behaviour cannot be defined as an infinite sum over all runs, although elements realised by such automata can be defined via stars of transition matrices. The next example shows that this is a case for *cycle-free weighted automata* over arbitrary coefficient semirings, as defined in [11]. This is the reason why automata over summation semirings, with semantics defined as in Section 3, *do not* incorporate all cycle-free weighted automata.

Example 8.1. Let S be a semiring and Σ an alphabet. A *cycle-free weighted automaton* over S and Σ [11] can be described as a finite automaton $\mathcal{A}$ over the summation semiring $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$ of Example 2.3[9] such that the transition matrix $\Delta = \Delta_{\mathcal{A}}$ of $\mathcal{A}$ satisfies the following property: if $\Delta^t = (d_{i,j}[t])_{n \times n}$ for all t in $\mathbb{N}$,

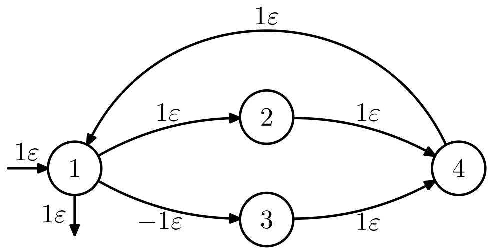

Figure 2: A cycle-free weighted automaton $\mathcal{A}$ with undefined behaviour over $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$.

then there is some s in $\mathbb{N}$ such that Δ^s is a matrix of proper series, i.e., $(d_{i,j}[s], \varepsilon) = 0$ for $i, j = 1, \ldots, n$. It is then easy to see that the matrix Δ^* is well defined over $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$.

However, the behaviour $\|\mathcal{A}\|$ of a cycle-free automaton $\mathcal{A}$ over S and Σ might be undefined. Figure 2 shows an example of such automaton $\mathcal{A}$. Clearly, there are infinitely many valid runs in $\mathcal{A}$ such that the corresponding series is 1; the sum for the behaviour is thus not locally finite and $\|\mathcal{A}\| = \bot$. On the other hand, it is easy to see that if $I = I_{\mathcal{A}}$ and $F = F_{\mathcal{A}}$ are vectors defined as in Section 3, then $I\Delta^* F = 1$.

The phenomenon just described is caused by the fact that condition *(ii)* of Definition 2.1 may not hold the other way round for general summation semirings, the summation semiring of formal power series with locally finite sums being a prominent counterexample. The condition *(ii)* guarantees that each well-defined infinite sum can be evaluated by parts – that is, it is possible to first sum over some subfamilies and next take the sum of results obtained. However, it may happen that a well-defined sum of well-defined sums becomes undefined after being "unwrapped" to a single infinite sum.

Evaluation of sums by parts is thus "stronger" than their "standard" evaluation. Thus, to incorporate cycle-free weighted automata into our framework, we shall now define an alternative semantics for finite automata over summation semirings, which we shall call partial evaluation semantics. This will correspond to evaluating the behaviour of an automaton $\mathcal{A}$ by parts, according to some partition of the language $\|\mathrm{temp}(\mathcal{A})\|$.[10] It is easy to see on Example 8.1 that different results can be obtained for different partitions.

In what follows, U denotes a universe of symbols (i.e., an infinite alphabet); we shall assume that $\|\mathrm{temp}(\mathcal{A})\| \subseteq U^*$ for each finite automaton $\mathcal{A}$ considered.

Definition 8.2. An *evaluation partition* is a partition $\mathcal{P} = (P_i \mid i \in I)$ of U^* such that $P_i \cap \Sigma^*$ is finite for each i in I and each finite alphabet $\Sigma \subseteq U$.

We shall now use the following notational shortcut: if S is a set, $(a_i \mid i \in I)$ a family in $\mathcal{F}(S)$, and $p(i)$ a predicate dependent on i, then we shall write $(a_i \mid i \in I;\ p(i))$ for the family $(a_i \mid i \in \{i' \in I \mid p(i')\})$. If moreover S is a summation semiring $(S, \mathcal{F}, \Phi)$ and $(a_i \mid i \in I;\ p(i))$ is in $\mathcal{F}$, then we shall write

$$\sum_{\substack{i \in I \\ p(i)}} a_i := \sum_{i \in \{i' \in I \mid p(i')\}} a_i.$$

Definition 8.3. Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, $\mathcal{P} = (P_i \mid i \in I)$ an evaluation partition, S' a subset of S containing the elements 0 and 1, and $\mathcal{A}$ a finite S'-automaton over S. If

$$\left(\sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap P_i} h[\mathcal{A}](w) \;\middle|\; i \in I;\ \|\mathrm{temp}(\mathcal{A})\| \cap P_i \neq \emptyset \right)$$

[9] That is: $\mathcal{F}$ consists of all *locally finite* families of series in $S\langle\langle\Sigma^*\rangle\rangle$.

[10] To be more precise, we shall require all parts to which $\|\mathrm{temp}(\mathcal{A})\|$ is decomposed to be finite. This is in order to avoid pathological situations, in which the partial sums used to evaluate the behaviour are themselves undefined.

is in $\mathcal{F}$,[11] we shall write

$$\|\mathcal{A}\|_{\mathcal{P}} := \sum_{\substack{i \in I \\ \|\mathrm{temp}(\mathcal{A})\| \cap P_i \neq \emptyset}} \sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap P_i} h[\mathcal{A}](w)$$

and call the semiring element $\|\mathcal{A}\|_{\mathcal{P}}$ the *behaviour of $\mathcal{A}$ according to partial evaluation semantics* given by $\mathcal{P}$. We shall write $\|\mathcal{A}\|_{\mathcal{P}} = \bot$ otherwise.

Remark 8.4. Let ID denote an evaluation partition $\mathsf{ID} = (\{w\} \mid w \in U^*)$. Then it is easy to see that

$$\|\mathcal{A}\| = \|\mathcal{A}\|_{\mathsf{ID}}$$

holds for every finite S'-automaton $\mathcal{A}$ over $(S, \mathcal{F}, \Phi)$.

We shall say that an evaluation partition $\mathcal{R} = (R_j \mid j \in J)$ is a *refinement* of an evaluation partition $\mathcal{P} = (P_i \mid i \in I)$ if there is a partition $(J_i \mid i \in I)$ of J such that $P_i = \bigcup_{j \in J_i} R_j$ for each i in I. We shall write $\mathcal{P} \sqsubseteq \mathcal{R}$ in that case.

Proposition 8.5. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and $\mathcal{A}$ a finite S'-automaton over S. If $\mathcal{P}, \mathcal{R}$ are evaluation partitions such that $\mathcal{P} \sqsubseteq \mathcal{R}$, then $\|\mathcal{A}\|_{\mathcal{P}} \neq \bot$ whenever $\|\mathcal{A}\|_{\mathcal{R}} \neq \bot$, in which case $\|\mathcal{A}\|_{\mathcal{P}} = \|\mathcal{A}\|_{\mathcal{R}}$. In particular, $\|\mathcal{A}\| \neq \bot$ implies $\|\mathcal{A}\|_{\mathcal{P}} = \|\mathcal{A}\|$ for each evaluation partition $\mathcal{P}$.*

Proof. Let $\mathcal{P} = (P_i \mid i \in I)$, $\mathcal{R} = (R_j \mid j \in J)$, and $\|\mathcal{A}\|_{\mathcal{R}} \neq \bot$. Then

$$\left(\sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap R_j} h[\mathcal{A}](w) \;\middle|\; j \in J;\ \|\mathrm{temp}(\mathcal{A})\| \cap R_j \neq \emptyset \right)$$

is in $\mathcal{F}$. Let $(J_i \mid i \in I)$ be a partition of J such that $P_i = \bigcup_{j \in J_i} R_j$ for each i in I. Let

$$J' := \{j \in J \mid \|\mathrm{temp}(\mathcal{A})\| \cap R_j \neq \emptyset\}$$

and for each i in I, let

$$J_i' := \{j \in J_i \mid \|\mathrm{temp}(\mathcal{A})\| \cap R_j \neq \emptyset\}.$$

Then $(J_i' \mid i \in I; \|\mathrm{temp}(\mathcal{A})\| \cap P_i \neq \emptyset)$ is a partition of J'. Moreover, $(R_j \mid j \in J_i)$ is clearly a partition of P_i for each i in I. Hence, $(\|\mathrm{temp}(\mathcal{A})\| \cap R_j \mid j \in J_i')$ is a partition of a *finite* set $\|\mathrm{temp}(\mathcal{A})\| \cap P_i$ for each i in I. It thus follows by conditions (ii) and (i) of Definition 2.1 that

$$\begin{aligned}
\|\mathcal{A}\|_{\mathcal{R}} &= \sum_{\substack{j \in J \\ \|\mathrm{temp}(\mathcal{A})\| \cap R_j \neq \emptyset}} \sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap R_j} h[\mathcal{A}](w) = \sum_{j \in J'} \sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap R_j} h[\mathcal{A}](w) = \\
&= \sum_{\substack{i \in I \\ \|\mathrm{temp}(\mathcal{A})\| \cap P_i \neq \emptyset}} \sum_{j \in J_i'} \sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap R_j} h[\mathcal{A}](w) = \sum_{\substack{i \in I \\ \|\mathrm{temp}(\mathcal{A})\| \cap P_i \neq \emptyset}} \sum_{w \in \|\mathrm{temp}(\mathcal{A})\| \cap P_i} h[\mathcal{A}](w) = \|\mathcal{A}\|_{\mathcal{P}},
\end{aligned}$$

all sums being over summable families. As a result, $\|\mathcal{A}\|_{\mathcal{P}}$ is well defined and equal to $\|\mathcal{A}\|_{\mathcal{R}}$.

For the second part of the proposition, it suffices to observe that $\mathcal{P} \sqsubseteq \mathsf{ID}$ holds for each evaluation partition $\mathcal{P}$, so that the claim follows by the first part of the proposition and by Remark 8.4. ☐

Let LEN denote an evaluation partition $\mathsf{LEN} = (U^t \mid t \in \mathbb{N})$. We shall now prove that $\|\mathcal{A}\|_{\mathsf{LEN}}$ is defined for a finite automaton $\mathcal{A}$ whenever the star of its transition matrix is. In particular, $\|\mathcal{A}\|_{\mathsf{LEN}}$ is defined over summation semirings of power series with locally finite sums for each cycle-free weighted automaton $\mathcal{A}$.

[11] Note that the sums constituting this family are well-defined, as $\|\mathrm{temp}(\mathcal{A})\| \cap P_i$ is finite by definition of evaluation partitions.

Proposition 8.6. *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and $\mathcal{A} = (\{1, \ldots, n\}, \iota, T, \tau)$ a finite S'-automaton over S with $n \geq 1$ in $\mathbb{N}$, $I_{\mathcal{A}} = I$, $\Delta_{\mathcal{A}} = \Delta$, and $F_{\mathcal{A}} = F$. If Δ^* is defined, then $\|\mathcal{A}\|_{\mathsf{LEN}}$ is defined as well and $\|\mathcal{A}\|_{\mathsf{LEN}} = I\Delta^*F$.*

Proof. Let $\Delta^t = (d_{i,j}[t])_{n\times n}$ for each $t \in \mathbb{N}$. Moreover, let E be the transition matrix of $\mathrm{temp}(\mathcal{A})$ and $E^t = (e_{i,j}[t])_{n\times n}$ for each $t \in \mathbb{N}$. Similarly as in the proof of Theorem 3.11, observe that

$$d_{i,j}[t] = \sum_{x \in e_{i,j}[t]} h[\mathcal{A}](x)$$

for $i, j = 1, \ldots, n$ and each t in $\mathbb{N}$. Moreover,

$$\|\mathrm{temp}(\mathcal{A})\| \cap U^t = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} (i,1)e_{i,j}[t](j,2)$$

for each t in $\mathbb{N}$, the unions being disjoint.

It then follows by conditions (i) to (iv) of Definition 2.1 that

$$\begin{aligned}
I\Delta^*F &= \sum_{i=1}^{n}\sum_{j=1}^{n} \iota(i)\left(\sum_{t\in\mathbb{N}} d_{i,j}[t]\right)\tau(j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t\in\mathbb{N}} \iota(i)d_{i,j}[t]\tau(j) = \\
&= \sum_{\substack{t\in\mathbb{N}\\ i,j\in\{1,\ldots,n\}}} \iota(i)d_{i,j}[t]\tau(j) = \sum_{t\in\mathbb{N}}\sum_{i=1}^{n}\sum_{j=1}^{n} \iota(i)d_{i,j}[t]\tau(j) = \\
&= \sum_{t\in\mathbb{N}}\sum_{i=1}^{n}\sum_{j=1}^{n} \iota(i)\left(\sum_{x\in e_{i,j}[t]} h[\mathcal{A}](x)\right)\tau(j) = \sum_{t\in\mathbb{N}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{w\in(i,1)e_{i,j}[t](j,2)} h[\mathcal{A}](w) = \\
&= \sum_{t\in\mathbb{N}}\sum_{w\in\|\mathrm{temp}(\mathcal{A})\|\cap U^t} h[\mathcal{A}](w) = \|\mathcal{A}\|_{\mathsf{LEN}},
\end{aligned}$$

all sums being over summable families. The proposition is proved. □

However, although slightly more powerful, partial evaluation semantics seems to be much less robust than the standard semantics for finite automata over summation semirings. In the preceding sections, we have defined templates for various models in a more or less arbitrary way, without taking much care of details that do not change the expressive power. Nevertheless, some more subtle properties become important under partial evaluation semantics. For instance, it is easy to see that the symbols added in the definition of template rational expressions in order to make the star unambiguous can result in same evaluation partition having a substantially different effect for finite automata and for rational expressions. The detailed examination of similar phenomena is left for further research.

9. Conclusion

The basic theory of rational elements of *summation semirings*, traditionally known as Σ-semirings [14], has been developed by defining the notions of finite automata, rational expressions, right-linear systems, and MSO logics over summation semirings and proving that these four models are equivalent in their expressive power. The proofs of expressive equivalences were based on the method of “homomorphic transitions” from semirings of formal languages to general summation semirings, making it possible to avoid “redoing” the usual equivalence proofs over formal languages in the more general setting of semirings.

Moreover, we have observed that the class of all summation semirings is incomparable with the class of all partial Conway semirings over ideals. The scope of the theory presented herein is thus incomparable with the scope of the theory developed by Bloom, Ésik, and Kuich [2, 12]. On the other hand, it seems that the settings truly relevant to automata theory can be captured using both approaches.

We have not addressed these questions of potential interest:

1. Is it possible to weaken the definition of summation semirings so that the main results proved in this article remain true and some new semirings (such as the one constructed in the proof of Proposition 4.5) are incorporated?
2. On the opposite side, is it possible to strengthen the definition of summation semirings so that the resulting class of semirings still remains a unifying framework for the theory of rational and algebraic semiring elements and so that some properties that do not hold over general summation semirings – such as closure of $\mathrm{Rat}(S', S)$ under well-defined star – are always satisfied?
3. What is the relation of well-formed formulae in MSO logics over summation semirings of formal power series and *syntactically restricted* formulae in weighted MSO logics of Droste and Gastin [4, 5]? In particular, both classes of formulae define precisely the class of rational power series. Is there some kind of natural correspondence between MSO formulae of both types?

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