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A Unifying Approach to Algebraic Systems over Semirings

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Abstract A fairly general definition of canonical solutions to algebraic systems over semirings is proposed. This is based on the notion of summation semirings, traditionally known as Σ-semirings, and on assigning unambiguous context-free languages to variables of each system. The presented definition applies to all algebraic systems over continuous or complete semirings and to all proper algebraic systems over power series semirings, for which it coincides with the usual definitions of their canonical solutions. As such, it unifies the approaches to algebraic systems over semirings studied in literature. An equally general approach is adopted to study pushdown automata, for which equivalence with algebraic systems is proved. Finally, the Chomsky-Schützenberger theorem is generalised to the setting of summation semirings.

Keywords Algebraic System · Semiring · Summation Semiring · Template

1 Introduction

Algebraic systems over formal power series semirings [14,15,17,19], essentially "isomorphic" to *weighted context-free grammars* [6,20], provide a natural weight-assigning extension to classical context-free grammars [11]. Instead of merely generating languages, algebraic systems define *formal power series* in several noncommutative variables [4,19]. That is, a weight given by an element of some semiring is assigned to each word. A correspondence between context-free grammars and algebraic systems thus is an "algebraic" counterpart of the correspondence between nondeterministic finite automata and weighted automata [4] on the level of rational phenomena.

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The vector of formal power series defined by an algebraic system is in general obtained as its unique canonical solution, for some specific meaning of "canonical". However, difficulties arise when trying to define such unique solutions both for all systems and for all semirings. For this reason, algebraic systems are usually studied under some restrictive conditions either on the underlying semiring of coefficients, or on the form of the system. In the former case, one usually deals with systems over semirings of formal power series, coefficients of which are taken from a *continuous semiring* [9, 14], or more generally from a *complete semiring* [6].[1] In the latter case, one is led to consider *proper algebraic systems* over arbitrary power series semirings [17, 19].

The first of the approaches described above can easily be generalised. One might consider algebraic systems over some subset S' of an arbitrary continuous (complete) semiring S, which is not necessarily a semiring of power series. Each such system defines a single vector of elements of S. For continuous semirings, this is the approach undertaken, e.g., in [9]. We shall adopt this generalisation as well – that is, we shall work with systems over semirings that might, but also might not, be semirings of formal power series.

This article aims to unify the approach based on arbitrary systems over continuous or complete (not necessarily power series) semirings with the approach via proper systems over arbitrary power series semirings by developing a common generalisation of both, and thus to eliminate the need for a choice when dealing with algebraic systems over semirings. Bloom, Ésik, and Kuich [1, 7] describe a unifying theory of this kind for finite automata over semirings. The present work can be seen as an attempt to come up with a comparably general theory for algebraic systems, though using a different approach.

The unifying approach introduced in this article will be based on semirings, which we shall call *summation semirings*. This is merely our alternative name for *Σ-semirings* of Hebisch and Weinert [10] (the reason for this change in terminology is to avoid the symbol Σ, which we use here for alphabets). These can be viewed as a generalisation of complete semirings, in which infinite sums are defined just for some families of elements called *summable families*.

We shall associate unambiguous context-free languages called *templates* with variables of each algebraic system (over some summation semiring S). Each template word (i.e., a word from a template language) for a variable y can be seen as capturing a structure of some derivation tree rooted in y in a "context-free grammar over S corresponding to the given system".[2] A template word w corresponding to a derivation tree of some semiring element s will have a property that s is a homomorphic image of w under a suitable homomorphism. Hence, a canonical solution to an algebraic system will be well defined if and only if the family of homomorphic images of its template words is summable for each variable, in which case the solution will be given by the vector of sums of these homomorphic images for respective variables.

[1] To be more precise, *weighted context-free grammars* over complete semirings are considered in [6]. These can nevertheless be viewed as algebraic systems.

[2] We shall dispense with a formally defined notion of weighted context-free grammars over semirings. Nevertheless, an idea of such grammars is useful for gaining intuition.

We shall see that canonical solutions are solutions in the usual sense and that their definition is consistent with typical definitions appearing in literature for systems over continuous and complete semirings and for proper systems over power series semirings.

An element of a summation semiring will be called *algebraic* if it is a well-defined first component of a "partial canonical solution" to some algebraic system – that is, of a canonical solution possibly containing some undefined components.

The definition of canonical solutions via templates has an advantage that several properties of context-free languages can simply be "lifted" to algebraic elements of a summation semiring. To demonstrate this process, we shall prove that algebraic systems over summation semirings are equivalent to suitably defined pushdown automata and extend the Chomsky-Schützenberger theorem to algebraic elements of a summation semiring.

The approach introduced in this article has a potential to be extended to structures more general than semirings, such as valuation monoids [6] or auxiliary weighting structures [13].

2 Preliminaries

A *family* $(a_i \mid i \in I)$ of elements of a set S, indexed by a set I, is a mapping $\varphi\colon I \to S$ such that $\varphi(i) = a_i$ for all i in I. A family $(a_i \mid i \in I)$ is finite, countably infinite, or infinite if the index set I has the respective property. More generally, $(a_i \mid i \in I)$ is of cardinality κ if I is of cardinality κ. We shall denote the class of all families of elements of S by $\mathcal{F}(S)$ and the class of all finite or countably infinite families of elements of S by $\mathcal{F}_\omega(S)$.

A *generalised partition* of a set I is a family $(I_j \mid j \in J)$ of subsets of I such that $I = \bigcup_{j\in J} I_j$ and $I_j \cap I_k = \emptyset$ for all j, k in J such that $j \neq k$. A *partition* of I is a generalised partition $(I_j \mid j \in J)$ of I such that $I_j \neq \emptyset$ for all j in J.

A *monoid* is a triple $(M, \cdot, 1)$, where M is a set, $\cdot$ is an associative binary operation on M, and 1 is a neutral element with respect to $\cdot$. A monoid $(M, \cdot, 1)$ is *commutative* if $\cdot$ is commutative. A *semiring* is a quintuple $(S, +, \cdot, 0, 1)$, where $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, $\cdot$ (multiplication in S) distributes over $+$ (addition in S) both from left and from right, and $0 \cdot a = a \cdot 0 = 0$ holds for all a in S. We shall often write S instead of $(S, +, \cdot, 0, 1)$.

A semiring S is (countably) *complete* [10,5] if sums over all (countably) infinite families of elements are defined in S. This is made precise by the following definition.

Definition 1 A *complete (countably complete) semiring* is a pair (S, Φ), where S is a semiring and $\Phi\colon \mathcal{F}(S) \to S$ ($\Phi\colon \mathcal{F}_\omega(S) \to S$) is a mapping assigning to each family $(a_i \mid i \in I)$ in $\mathcal{F}(S)$ (in $\mathcal{F}_\omega(S)$) a semiring element

$$\Phi(a_i \mid i \in I) =: \sum_{i\in I} a_i$$

so that the following conditions are satisfied:

($c1$) Let n be in $\mathbb{N}$ and $I = \{i_1, \ldots, i_n\}$ a finite set with n elements. Then

$$\sum_{i \in I} a_i = a_{i_1} + \ldots + a_{i_n}.$$

($c2$) Let $(a_i \mid i \in I)$ be in $\mathcal{F}(S)$ (in $\mathcal{F}_\omega(S)$) and $(I_j \mid j \in J)$ a generalised partition (a finite or countably infinite generalised partition) of I. Then

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

($c3$) Let $(a_i \mid i \in I)$ be in $\mathcal{F}(S)$ (in $\mathcal{F}_\omega(S)$) and a in S. Then

$$a \cdot \left(\sum_{i \in I} a_i \right) = \sum_{i \in I} (a \cdot a_i) \qquad \text{and} \qquad \left(\sum_{i \in I} a_i \right) \cdot a = \sum_{i \in I} (a_i \cdot a).$$

Let $(X, \leq)$ be a partially ordered set. A *directed set* [3] in X is a subset D of X that contains at least one upper bound of $\{x, y\}$ for each x, y in D. A *complete partially ordered set* (CPO) [3] is a partially ordered set $(X, \leq)$ with a bottom element (the least element of X) and a least upper bound $\sup D$ for each directed subset D of X. A *continuous semiring* [5,8] is a semiring $(S, +, \cdot, 0, 1)$ with a partial order $\leq$ on S such that $(S, \leq)$ is a CPO with bottom element 0 and such that $a + \sup D = \sup(a + D)$, $a \cdot (\sup D) = \sup(a \cdot D)$, and $(\sup D) \cdot a = \sup(D \cdot a)$ holds for each a in S and each directed subset D of S. One might define infinite sums in a continuous semiring S by

$$\sum_{i \in I} a_i = \sup \left\{ \sum_{i \in F} a_i \;\middle|\; F \subseteq I, F \text{ finite} \right\}.$$

It is easy to check that S forms a complete semiring with infinite sums defined like this [5]. Hence, each continuous semiring is complete.

Let S be a semiring, $Y = \{y_1, \ldots, y_t\}$ a nonempty finite alphabet of variables, and $S \cap Y^+ = \emptyset$. Essentially following [9], let us define a *representation*[3] of a *semiring-polynomial* over S and Y to be a formal sum $p = m_1 + \ldots + m_k$, where $k \geq 1$ is in $\mathbb{N}$ and $m_1, \ldots, m_k$ are representations of semiring-monomials over S and Y. Here, a *representation* of a *semiring-monomial* over S and Y is an alternating sequence $m = (a_0, y_{i_1}, a_1, \ldots, a_{r-1}, y_{i_r}, a_r)$ with r in $\mathbb{N}$, coefficients $a_0, \ldots, a_r$ in S, and variable indices $i_1, \ldots, i_r$ in $\{1, \ldots, t\}$. We shall usually write $a_0 y_{i_1} a_1 \ldots a_{r-1} y_{i_r} a_r$ instead of $(a_0, y_{i_1}, a_1, \ldots, a_{r-1}, y_{i_r}, a_r)$.

A representation of a S'-monomial over S and Y, where S' is some subset of S, is a representation of a semiring-monomial $a_0 y_{i_1} a_1 \ldots a_{r-1} y_{i_r} a_r$ such that

[3] Semiring-polynomials can be introduced as a specialisation of the notion of polynomials over a universal algebra [16]. With this definition, a semiring-polynomial can be seen as a congruence class of a suitable congruence defined on the algebra of *representations* of semiring-polynomials as defined below. However, let us stress that this distinction is unimportant for our purposes, since similarly as in [9], we shall only be interested in mappings induced by (representations of) semiring-polynomials.

the coefficients $a_0, \dots, a_r$ are in S'. A representation of a S'-polynomial over S and Y is a formal sum of representations of S'-monomials. We shall call the representations of semiring-monomials $m_1, \dots, m_k$ such that $p = m_1 + \dots + m_k$ the *terms* of p and write $\tau(p)$ for the set $\{m_1, \dots, m_k\}$.

Sums and products of representations of semiring-polynomials can be defined in an obvious way.[4] A *monomial function* induced by a representation of a semiring-monomial $m = a_0 y_{i_1} a_1 \dots a_{r-1} y_{i_r} a_r$ over S and $Y = \{y_1, \dots, y_t\}$ is a mapping $\overline{m}\colon S^t \to S$ defined by $\overline{m}(s_1, \dots, s_t) = a_0 s_{i_1} a_1 \dots a_{r-1} s_{i_r} a_r$ for all $s_1, \dots, s_t$ in S. A *polynomial function* [9] induced by $p = m_1 + \dots + m_k$, where k is in $\mathbb{N}$ and $m_1, \dots, m_k$ are representations of semiring-monomials, is a mapping $\overline{p} = \overline{m_1} + \dots + \overline{m_k}$. We shall usually write p instead of $\overline{p}$.

Let S be a semiring and Σ a nonempty finite alphabet. A *formal power series* over S and Σ is a mapping $r\colon \Sigma^* \to S$. The value $r(w)$ of r on a word w in Σ^* is usually denoted by (r, w), and the series r itself is written as

$$r = \sum_{w \in \Sigma^*} (r, w) w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle \Sigma^* \rangle\rangle$.

A formal power series r in $S\langle\langle \Sigma^* \rangle\rangle$ is *proper* if $(r, \varepsilon) = 0$. The set of all proper series over S and Σ is denoted by $S\langle\langle \Sigma^+ \rangle\rangle$.

Let r_1 and r_2 be in $S\langle\langle \Sigma^* \rangle\rangle$. The *sum* of r_1 and r_2 is a series $r_1 + r_2$ in $S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_1 + r_2, w) = (r_1, w) + (r_2, w)$ for all w in Σ^*. The *Cauchy product* of r_1 and r_2 is a series $r_1 \cdot r_2$ in $S\langle\langle \Sigma^* \rangle\rangle$ such that

$$(r_1 \cdot r_2, w) = \sum_{\substack{u, v \in \Sigma^* \\ uv = w}} (r_1, u)(r_2, v)$$

for all w in Σ^*. It is easy to prove that $S\langle\langle \Sigma^* \rangle\rangle$ constitutes a semiring with these two operations [4].

A family $(r_i \mid i \in I)$ of formal power series in $S\langle\langle \Sigma^* \rangle\rangle$ is *locally finite* if the set $I(w) = \{i \in I \mid (r_i, w) \neq 0\}$ is finite for all w in Σ^*.

If r in $S\langle\langle \Sigma^* \rangle\rangle$ is a series, then the *support* of r, denoted by $\mathrm{supp}(r)$, is the language of all w in Σ^* such that $(r, w) \neq 0$. A *polynomial* over S and Σ is a series r in $S\langle\langle \Sigma^* \rangle\rangle$ such that $\mathrm{supp}(r)$ is finite. The set of all polynomials over S and Σ is denoted by $S\langle \Sigma^* \rangle$.

Let S be a commutative[5] semiring and Σ a nonempty finite alphabet. An *algebraic system* over S and Σ [17,19] can be defined to be a pair $(\mathbf{y}, \mathbf{p})$, where $\mathbf{y} = (y_1, \dots, y_t)^T$ is a vector of variables such that $t \geq 1$ and $\Sigma \cap Y = \emptyset$ holds for $Y = \{y_1, \dots, y_t\}$, and where $\mathbf{p} = (p_1, \dots, p_t)^T$ is a vector of polynomials from $S\langle (\Sigma \cup Y)^* \rangle$. One usually writes $\mathbf{y} = \mathbf{p}$ instead of $(\mathbf{y}, \mathbf{p})$. A system $\mathbf{y} = \mathbf{p}$

[4] Note that the resulting algebra is not a semiring. On the other hand, the algebra of semiring-polynomials, which can be obtained as a factor algebra of the algebra of their representations, constitutes a semiring [16].

[5] Commutativity of S is a commonplace assumption when dealing with algebraic systems over $S\langle\langle \Sigma^* \rangle\rangle$ [17,19]. However, let us note that it is not strictly necessary [17] and that the approach presented later in this article subsumes the noncommutative case as well.

with $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$ is *proper* [17,19] if $(p_i, \varepsilon) = 0$ holds for $i = 1, \ldots, t$ and $(p_i, y_j) = 0$ holds for $i, j = 1, \ldots, t$. This means that a system is proper if it corresponds to an ε-free grammar without chain rules.

A *solution* to an algebraic system $\mathbf{y} = \mathbf{p}$ over S and Σ with $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$ is a vector $\mathbf{r} = (r_1, \ldots, r_t)^T$ of series from $S\langle\langle \Sigma^* \rangle\rangle$ such that

$$r_i = \sum_{w \in \mathrm{supp}(p_i)} (p_i, w) h_{\mathbf{r}}(w)$$

holds for $i = 1, \ldots, t$ and a monoid homomorphism $h_{\mathbf{r}} \colon (\Sigma \cup Y)^* \to (S\langle\langle \Sigma^* \rangle\rangle, \cdot)$ defined by $h_{\mathbf{r}}(y_j) = r_j$ for $j = 1, \ldots, t$ and $h_{\mathbf{r}}(c) = c$ for all c in Σ.

It is a well known fact that each proper algebraic system $\mathbf{y} = \mathbf{p}$ over S and Σ has a *unique proper solution*, i.e., a unique solution such that all its components are proper series [17,19]. Here, let us just note that one way to establish this fact is to define a metric on $S\langle\langle \Sigma^* \rangle\rangle$ in a usual way [18] by

$$d(r_1, r_2) = \begin{cases} 2^{-\min\{|w| \ \mid \ w \in \Sigma^*;\ (r_1, w) \neq (r_2, w)\}} & \text{if } r_1 \neq r_2 \\ 0 & \text{if } r_1 = r_2 \end{cases}$$

for all r_1, r_2 in $S\langle\langle \Sigma^* \rangle\rangle$ and to observe that the subspace $(S\langle\langle \Sigma^+ \rangle\rangle, d)$ of the metric space $(S\langle\langle \Sigma^* \rangle\rangle, d)$ is complete. The metric d can then be extended to $(S\langle\langle \Sigma^* \rangle\rangle)^t$ for all integers $t \geq 1$ by taking a maximum over all t components, again yielding a complete subspace $((S\langle\langle \Sigma^+ \rangle\rangle)^t, d)$. It is easy to show that the "iteration" of each proper algebraic system is a contraction on $((S\langle\langle \Sigma^+ \rangle\rangle)^t, d)$ for some t, implying existence of a unique proper solution by the Banach fixed point theorem.

3 Summation Semirings

The unifying approach to algebraic systems developed in this article will be based on semirings that generalise complete semirings and countably complete semirings in that infinite sums are not required to be defined for all infinite or countably infinite families of elements, but just for so called *summable families*. Such semirings are called *Σ-semirings* by Hebisch and Weinert [10]. To avoid possible confusion regarding the symbol Σ when it comes to alphabets, we shall use the term *summation semirings* instead.

Definition 2 A *summation semiring* is a triple $(S, \mathcal{F}, \Phi)$, where S is a semiring, $\mathcal{F}$ is a nonempty subclass of $\mathcal{F}(S)$ consisting of *summable families*, and $\Phi \colon \mathcal{F} \to S$ is a mapping assigning to each family $(a_i \mid i \in I)$ in $\mathcal{F}$ a semiring element

$$\Phi(a_i \mid i \in I) =: \sum_{i \in I} a_i$$

so that the following conditions are satisfied:

(i) Let n be in $\mathbb{N}$ and $I = \{i_1, \ldots, i_n\}$ a finite set with n elements. Then each $(a_i \mid i \in I)$ in $\mathcal{F}(S)$ is in $\mathcal{F}$ and

$$\sum_{i \in I} a_i = a_{i_1} + \ldots + a_{i_n}.$$

(ii) Let $(a_i \mid i \in I)$ be in $\mathcal{F}$ and $(I_j \mid j \in J)$ a generalised partition of I such that $|J| \leq \kappa$ for some κ that is a cardinality of at least one family in $\mathcal{F}$. Then $(a_i \mid i \in I_j)$ is in $\mathcal{F}$ for all j in J, $\left(\sum_{i \in I_j} a_i \;\middle|\; j \in J\right)$ is in $\mathcal{F}$, and

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

(iii) Let $(a_i \mid i \in I)$ be in $\mathcal{F}(S)$, J a finite set, and $(I_j \mid j \in J)$ a generalised partition of I such that $(a_i \mid i \in I_j)$ is in $\mathcal{F}$ for all j in J. Then $(a_i \mid i \in I)$ is in $\mathcal{F}$ as well and, as a consequence of (ii) and (i),

$$\sum_{i \in I} a_i = \sum_{j \in J} \left(\sum_{i \in I_j} a_i \right).$$

(iv) Let $(a_i \mid i \in I)$ and $(b_j \mid j \in J)$ be in $\mathcal{F}$. Then $(a_i \cdot b_j \mid (i,j) \in I \times J)$ is in $\mathcal{F}$ as well and

$$\left(\sum_{i \in I} a_i \right) \cdot \left(\sum_{j \in J} b_j \right) = \sum_{(i,j) \in I \times J} (a_i \cdot b_j).$$

Example 1 Each complete semiring (S, Φ) constitutes a summation semiring $(S, \mathcal{F}, \Phi)$, where $\mathcal{F} = \mathcal{F}(S)$. Indeed, the condition $(c1)$ of Definition 1 implies the condition (i) of Definition 2, the condition $(c2)$ of Definition 1 implies the condition (ii) of Definition 2, the condition (iii) of Definition 2 is trivially satisfied, and the condition (iv) of Definition 2 is a consequence of conditions $(c1)$, $(c2)$, and $(c3)$ of Definition 1: as $I \times J$ admits a generalised partition

$$I \times J = \bigcup_{i \in I} \{i\} \times J = \bigcup_{i \in I} \bigcup_{j \in J} \{i\} \times \{j\},$$

it follows that

$$\sum_{(i,j) \in I \times J} (a_i \cdot b_j) = \sum_{i \in I} \sum_{(i',j) \in \{i\} \times J} (a_{i'} \cdot b_j) = \sum_{i \in I} \sum_{j \in J} \sum_{(i',j') \in \{i\} \times \{j\}} (a_{i'} \cdot b_{j'}) =$$

$$= \sum_{i \in I} \sum_{j \in J} (a_i \cdot b_j) = \sum_{i \in I} \left(a_i \cdot \sum_{j \in J} b_j \right) = \left(\sum_{i \in I} a_i \right) \left(\sum_{j \in J} b_j \right).$$

Similarly, each countably complete semiring (S, Φ) constitutes a summation semiring $(S, \mathcal{F}, \Phi)$, where $\mathcal{F}$ consists of all finite or countably infinite families in $\mathcal{F}(S)$, i.e., $\mathcal{F} = \mathcal{F}_\omega(S)$. This follows in the same way as above, given the fact that the sets J and I_j for j in J of condition (ii), the set I of condition (iii), and the set $I \times J$ of condition (iv) are all finite or countably infinite.

Example 2 Each semiring $S\langle\langle\Sigma^*\rangle\rangle$ of formal power series constitutes a summation semiring $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$, where $\mathcal{F}$ is the class of all locally finite families of power series in $\mathcal{F}(S\langle\langle\Sigma^*\rangle\rangle)$ and $\Phi(r_i \mid i \in I) = \sum_{i \in I} r_i = r$ for all $(r_i \mid i \in I)$ in $\mathcal{F}$, where r is a series such that $(r, w) = \sum_{i \in I(w)} (r_i, w)$ for each w in Σ^*.

Indeed, if $I = \{i_1, \ldots, i_n\}$ for some n in $\mathbb{N}$, then each family $(r_i \mid i \in I)$ in $\mathcal{F}(S\langle\langle\Sigma^*\rangle\rangle)$ is locally finite, and thus in $\mathcal{F}$. Moreover, for each w in Σ^*,

$$\left(\sum_{i \in I} r_i, w\right) = \sum_{i \in I(w)} (r_i, w) = \sum_{i \in I(w)} (r_i, w) + \sum_{i \in I - I(w)} 0 = \\ = \sum_{i \in I} (r_i, w) = (r_{i_1} + \ldots + r_{i_n}, w).$$

Hence, $\sum_{i \in I} r_i = r_{i_1} + \ldots + r_{i_n}$ and the condition (i) of Definition 2 is satisfied.

Next, let $(r_i \mid i \in I)$ be a locally finite family in $\mathcal{F}$ and $(I_j \mid j \in J)$ a generalised partition of I.[6] Then one has $I_j(w) = I_j \cap I(w) \subseteq I(w)$ for each j in J and w in Σ^*, implying that $(r_i \mid i \in I_j)$ is locally finite and thus in $\mathcal{F}$. In addition, the set $I(w)$ admits a partition $I(w) = \bigcup_{j \in J[w]} I_j(w)$, where $J[w]$ is a *finite* set $J[w] := \{j \in J \mid I_j(w) \neq \emptyset\}$; clearly, $J(w) \subseteq J[w]$ holds for

$$J(w) = \left\{ j \in J \;\middle|\; \left(\sum_{i \in I_j} r_i, w\right) \neq 0 \right\}.$$

Thus, $\left(\sum_{i \in I_j} r_i \;\middle|\; j \in J\right)$ is locally finite. Moreover, for w in Σ^* fixed,

$$\left(\sum_{j \in J} \left(\sum_{i \in I_j} r_i\right), w\right) = \sum_{j \in J(w)} \left(\sum_{i \in I_j} r_i, w\right) = \\ = \sum_{j \in J(w)} \left(\sum_{i \in I_j} r_i, w\right) + \sum_{j \in J[w] - J(w)} 0 = \\ = \sum_{j \in J[w]} \left(\sum_{i \in I_j} r_i, w\right) = \sum_{j \in J[w]} \sum_{i \in I_j(w)} (r_i, w) = \\ = \sum_{i \in I(w)} (r_i, w) = \left(\sum_{i \in I} r_i, w\right).$$

This implies that

$$\sum_{j \in J} \left(\sum_{i \in I_j} r_i\right) = \sum_{i \in I} r_i$$

and the condition (ii) of Definition 2 is satisfied.

[6] This generalised partition can be of any cardinality, as $(0 \mid i \in I')$ is in $\mathcal{F}$ for any I'.

For (iii), suppose that $(r_i \mid i \in I)$ is in $\mathcal{F}(S\langle\langle \Sigma^* \rangle\rangle)$ and $(I_j \mid j \in J)$ is a generalised partition of I such that J is a finite set and $(r_i \mid i \in I_j)$ is locally finite for each j in J. Let w be in Σ^*. Then $I_j(w)$ is finite for each j in J and $I(w) = \bigcup_{j \in J} I_j(w)$. Hence, $I(w)$ is finite. As w in Σ^* is arbitrary, $(r_i \mid i \in I)$ is locally finite and thus in $\mathcal{F}$ – the condition (iii) of Definition 2 is satisfied.

Finally, let $(r_i \mid i \in I)$ and $(s_j \mid j \in J)$ be locally finite families in $\mathcal{F}$. Let w in Σ^* be fixed. Then $(r_i \cdot s_j, w)$ can be nonzero only if there are u, v in Σ^* such that $w = uv$ and such that i is in $I(u)$ and j is in $J(v)$. Finiteness of w and of the sets $I(u)$ and $J(v)$ then implies that there are at most finitely many pairs (i, j) in $I \times J$ such that $(r_i \cdot s_j, w) \neq 0$. As this holds for all w in Σ^*, the family $(r_i \cdot s_j \mid (i, j) \in I \times J)$ is locally finite. Moreover, for each w in Σ^*,

$$\left(\sum_{(i,j) \in I \times J} (r_i \cdot s_j), w \right) = \sum_{(i,j) \in (I \times J)(w)} \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} (r_i, u)(s_j, v) =$$

$$= \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} \sum_{i \in I(u)} \sum_{j \in J(v)} (r_i, u)(s_j, v) =$$

$$= \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} \left(\left(\sum_{i \in I(u)} (r_i, u) \right) \left(\sum_{j \in J(v)} (s_j, v) \right) \right) =$$

$$= \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} \left(\left(\sum_{i \in I} r_i, u \right) \left(\sum_{j \in J} s_j, v \right) \right) =$$

$$= \left(\left(\sum_{i \in I} r_i \right) \left(\sum_{j \in J} s_j \right), w \right).$$

As a consequence,

$$\sum_{(i,j) \in I \times J} (r_i \cdot s_j) = \left(\sum_{i \in I} r_i \right) \left(\sum_{j \in J} s_j \right)$$

and the condition (iv) of Definition 2 is satisfied as well.

4 Algebraic Systems over Summation Semirings

We shall now introduce the notion of algebraic systems over summation semirings. The definition of algebraic systems themselves poses no serious problem – in fact, one could use the same definition for systems over an arbitrary semiring. The definition of a solution poses no problem as well. The delicate part of the following considerations is the definition of the *canonical* solution, which is needed in order for a system to define a uniquely determined vector of elements of the underlying semiring S. This is the place, where we shall make use of the fact that S is a summation semiring.

Definition 3 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing 0 and 1. An S'*-algebraic system* over S is a pair $(\mathbf{y}, \mathbf{p})$, where $\mathbf{y} = (y_1, \ldots, y_t)^T$ is a vector of variables with $t \geq 1$, where $S \cap Y^+ = \emptyset$ holds for $Y = \{y_1, \ldots, y_t\}$, and where $\mathbf{p} = (p_1, \ldots, p_t)^T$ is a vector of representations of S'-polynomials over S and Y. We shall usually write $\mathbf{y} = \mathbf{p}$ for $(\mathbf{y}, \mathbf{p})$.

Definition 4 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and $\mathbf{y} = \mathbf{p}$ an S'-algebraic system over S, where $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$. A *solution* to $\mathbf{y} = \mathbf{p}$ is a vector $(s_1, \ldots, s_t)^T$ in S^t such that $s_j = p_j(s_1, \ldots, s_t)$ holds for $j = 1, \ldots, t$.

We have defined a solution to an S'-algebraic system over S, but we have no guarantee of its uniqueness. Our next aim therefore is to define a *canonical* solution to an S'-algebraic system, which, if it exists, uniquely determines a vector of elements of S defined by the system. In order to define canonical solutions, we shall first assign to each S'-algebraic system over S an algebraic system over a semiring of formal languages over a suitable alphabet, which we shall call a *template system*. This we shall obtain from the original system by replacing each monomial representation $m = a_0 y_{i_1} a_1 \ldots a_{r-1} y_{i_r} a_r$ appearing in the system by $m' = c_{m,0} y_{i_1} c_{m,1} \ldots c_{m,r-1} y_{i_r} c_{m,r}$, where $c_{m,0}, \ldots, c_{m,r}$ are symbols of the underlying alphabet.[7] The template system thus corresponds to a context-free grammar in the classical language-generating sense. Note that this grammar is always unambiguous (it is "almost" an s-grammar [12]).

Definition 5 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and $\mathbf{y} = \mathbf{p}$ an S'-algebraic system over S, where $\mathbf{y} = (y_1, \ldots, y_t)^T$, $\mathbf{p} = (p_1, \ldots, p_t)^T$, and

$$m = a_{m,0} y_{i(m,1)} a_{m,1} \ldots a_{m,r_m-1} y_{i(m,r_m)} a_{m,r_m}$$

for $j = 1, \ldots, t$ and for each m in $\tau(p_j)$, so that r_m is in $\mathbb{N}$, $a_{m,0}, \ldots, a_{m,r_m}$ are in S', and $i(m,1), \ldots, i(m,r_m)$ are in $\{1, \ldots, t\}$. Let $\Sigma = \Sigma(\mathbf{y}, \mathbf{p})$ be an alphabet defined by

$$\Sigma = \bigcup_{j=1}^{t} \bigcup_{m \in \tau(p_j)} \{c_{m,0}, \ldots, c_{m,r_m}\}.$$

Let us denote by 2_1^Σ the set of all singleton subsets of Σ. A *template system* corresponding to $\mathbf{y} = \mathbf{p}$ is a $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-algebraic[8] system $\mathbf{y} = \mathrm{temp}(\mathbf{p})$ over 2^{Σ^*} such that $\mathrm{temp}(\mathbf{p}) = (\mathrm{temp}(p_1), \ldots, \mathrm{temp}(p_t))^T$, where

$$\mathrm{temp}(p_j) = \sum_{m \in \tau(p_j)} c_{m,0} y_{i(m,1)} c_{m,1} \ldots c_{m,r_m-1} y_{i(m,r_m)} c_{m,r_m}$$

[7] More precisely, we should write $m' = \{c_{m,0}\} y_1 \{c_{m,1}\} \ldots \{c_{m,r-1}\} y_r \{c_{m,r}\}$. However, we follow here the common practice of identifying a singleton set $\{c\}$ with c itself.

[8] The polynomials of a template system will in fact all be 2_1^Σ-polynomials. However, the sets $\emptyset$ and $\{\varepsilon\}$ are the zero and the unity in the semiring 2^{Σ^*}, so they have to be included in order to satisfy the technical condition imposed in Definition 3.

for $j = 1, \ldots, t$. Let $(\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1, \ldots, \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_t)^T$ be the least solution to the system $\mathbf{y} = \mathrm{temp}(\mathbf{p})$; we shall call the language $\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j$ the *template* corresponding to y_j, for $j = 1, \ldots, t$. Let $h[\mathbf{y}, \mathbf{p}]\colon \Sigma^* \to (S, \cdot)$ be a monoid homomorphism given by $h[\mathbf{y}, \mathbf{p}](c_{m,i}) = a_{m,i}$ for each monomial m in $\bigcup_{j=1}^{t} \tau(p_j)$ and $i = 1, \ldots, r_m$. If $(h[\mathbf{y}, \mathbf{p}](w) \mid w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j)$ is in $\mathcal{F}$ for some j in $\{1, \ldots, t\}$, we shall say that the j-th component $\|\mathbf{y} = \mathbf{p}\|_j$ of the canonical solution to $\mathbf{y} = \mathbf{p}$ is defined and write

$$\|\mathbf{y} = \mathbf{p}\|_j := \sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j} h[\mathbf{y}, \mathbf{p}](w).$$

If the element $\|\mathbf{y} = \mathbf{p}\|_j$ is defined for $j = 1, \ldots, t$, we shall say that the vector $(\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t)^T$ is the *canonical solution* to $\mathbf{y} = \mathbf{p}$. Otherwise we shall say that the canonical solution is undefined.

Remark 1 Let us note that the above definition in fact contains a hidden (but obvious) proposition: if $(S, \mathcal{F}, \Phi)$ is the semiring of formal languages 2^{Σ^*} over some alphabet Σ and S' is the set $2_1^{\Sigma} \cup \{\emptyset, \{\varepsilon\}\}$, then canonical solutions to S'-algebraic systems over S coincide with their least solutions. This justifies the use of the common $\|\cdot\|_j$-notation both for the canonical solution to the original system and for the least solution to the template system.

Remark 2 Note that the idea behind Definition 5 is similar to the idea behind the homomorphism theorem for weighted context-free grammars in Greibach normal form proved by Stanat [20] and later in a more general setting by Droste and Vogler [6]. However, the unambiguous context-free languages used in [20, 6] are defined in a slightly different way than our templates, which necessitates the Greibach normal form requirement.

Definition 6 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing 0 and 1. An element a of the semiring S is S'*-algebraic* over S if there is an S'-algebraic system $\mathbf{y} = \mathbf{p}$ over S such that $\|\mathbf{y} = \mathbf{p}\|_1$ is defined and $a = \|\mathbf{y} = \mathbf{p}\|_1$. (Note that the canonical solution to $\mathbf{y} = \mathbf{p}$ might be undefined, although $\|\mathbf{y} = \mathbf{p}\|_1$ is defined.)

The following simple property of systems will be useful in what follows. Intuitively, it states that applying the template system $\mathbf{y} = \mathrm{temp}(\mathbf{p})$ (i.e., iterating $\mathrm{temp}(\mathbf{p})$) and then applying the homomorphism $h[\mathbf{y}, \mathbf{p}]$ is – under some reasonable circumstances – equivalent to first applying the homomorphism $h[\mathbf{y}, \mathbf{p}]$ and then applying the system $\mathbf{y} = \mathbf{p}$ over the summation semiring.

Lemma 1 *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and $\mathbf{y} = \mathbf{p}$ an S'-algebraic system over S, where $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$. Let $L_1, \ldots, L_t$ be languages over the alphabet $\Sigma(\mathbf{y}, \mathbf{p})$ such that $L_i \subseteq \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_i$ and $(h[\mathbf{y}, \mathbf{p}](w) \mid w \in L_i)$ is in $\mathcal{F}$ for $i = 1, \ldots, t$. Then $(h[\mathbf{y}, \mathbf{p}](w) \mid w \in \mathrm{temp}(p_j)(L_1, \ldots, L_t))$ is in $\mathcal{F}$ for $j = 1, \ldots, t$ as well and*

$$\sum_{w \in \mathrm{temp}(p_j)(L_1, \ldots, L_t)} h[\mathbf{y}, \mathbf{p}](w) = p_j \left(\sum_{w \in L_1} h[\mathbf{y}, \mathbf{p}](w), \ldots, \sum_{w \in L_t} h[\mathbf{y}, \mathbf{p}](w) \right).$$

Proof As $\mathbf{y} = \mathrm{temp}(\mathbf{p})$ corresponds to an unambiguous grammar and as $L_i \subseteq \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_i$ for $i = 1, \ldots, t$, all unions in $\mathrm{temp}(p_1), \ldots, \mathrm{temp}(p_t)$ are disjoint after substituting $L_1, \ldots, L_t$ for $y_1, \ldots, y_t$, and all products are unambiguous (for definition, see, e.g., Sakarovitch [18], p. 228). The claim then follows by the conditions (*iii*) and (*iv*) imposed on summation semirings in Definition 2 and by an easy structural induction on $p_1, \ldots, p_t$. □

We have defined canonical solutions to algebraic systems over summation semirings, but so far we have not proved that they are solutions in the sense of Definition 4. We shall now establish this claim.

Proposition 1 *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing the elements 0 and 1, and $\mathbf{y} = \mathbf{p}$ an S'-algebraic system over S, where $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$. Suppose that the canonical solution $(\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t)^T$ to $\mathbf{y} = \mathbf{p}$ is defined. Then it is a solution to $\mathbf{y} = \mathbf{p}$.*

Proof As $(\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1, \ldots, \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_t)^T$ is a solution to the system $\mathbf{y} = \mathrm{temp}(\mathbf{p})$, it follows that

$$\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j = \mathrm{temp}(p_j)(\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1, \ldots, \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_t)$$

for $j = 1, \ldots, t$. As a result, Lemma 1 gives us

$$\sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j} h[\mathbf{y}, \mathbf{p}](w) = \\ = p_j \left(\sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1} h[\mathbf{y}, \mathbf{p}](w), \ldots, \sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_t} h[\mathbf{y}, \mathbf{p}](w) \right),$$

which rewrites to

$$\|\mathbf{y} = \mathbf{p}\|_j = p_j(\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t).$$

Hence, $(\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t)^T$ is a solution to $\mathbf{y} = \mathbf{p}$. □

One is usually interested in *least* solutions when dealing with algebraic systems over *continuous* semirings. We shall now prove that our canonical solutions coincide with least solutions for summation semirings obtained from continuous semirings by defining infinite sums for all families of elements in the usual way. This means that the theory developed herein is consistent with the well explored theory of algebraic systems over continuous semirings [9, 14].

Proposition 2 *Let S be a continuous semiring with partial ordering $\leq$, and let $(S, \mathcal{F}, \Phi)$ be a summation semiring such that $\mathcal{F} = \mathcal{F}(S)$ and*

$$\Phi(a_i \mid i \in I) = \sum_{i \in I} a_i = \sup \left\{ \sum_{i \in F} a_i \;\middle|\; F \subseteq I, F \text{ finite} \right\}$$

for all families $(a_i \mid i \in I)$ in $\mathcal{F}$. Let S' be a subset of S containing 0 and 1, and let $\mathbf{y} = \mathbf{p}$ be an S'-algebraic system over S, where $\mathbf{y} = (y_1, \ldots, y_t)^T$ and $\mathbf{p} = (p_1, \ldots, p_t)^T$. Then the canonical solution $(\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t)^T$ is defined and equal to the least solution of $\mathbf{y} = \mathbf{p}$ with respect to $\leq$.

Proof Let X be a set and $\mathbf{f} = (f_1, \ldots, f_n)^T$ be a vector such that $n \geq 1$ is in $\mathbb{N}$ and $f_1, \ldots, f_n \colon X^n \to X$ are mappings. Let $x_1, \ldots, x_n$ be in X. Then we shall write $\mathbf{f}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))^T$. Let us also write $\mathbf{f}^{(0)}(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^T$ and $\mathbf{f}^{(k+1)} = \mathbf{f}(\mathbf{f}^{(k)}(x_1, \ldots, x_n))$ for k in $\mathbb{N}$.

Now, it is well known [9] that the least solution of $\mathbf{y} = \mathbf{p}$ is given by $\sup\{\mathbf{p}^{(k)}(0, \ldots, 0) \mid k \in \mathbb{N}\}$. For all k in $\mathbb{N}$, let us denote

$$\left(T_1^{(k)}, \ldots, T_t^{(k)}\right) := \mathrm{temp}(\mathbf{p})^{(k)}(\emptyset, \ldots, \emptyset).$$

By an easy induction using Lemma 1, it follows that

$$\mathbf{p}^{(k)}(0, \ldots, 0) = \left(\sum_{w \in T_1^{(k)}} h[\mathbf{y}, \mathbf{p}](w), \ldots, \sum_{w \in T_t^{(k)}} h[\mathbf{y}, \mathbf{p}](w)\right)^T$$

holds for all k in $\mathbb{N}$. For $j = 1, \ldots, t$, we have $T_j^{(k)} \subseteq T_j^{(k+1)}$ for all k in $\mathbb{N}$ and $\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j = \bigcup_{k \in \mathbb{N}} T_j^{(k)}$. Moreover, as S is continuous, $a \leq a + b$ holds for all a, b in S. As a result, it follows by the definition of infinite sums that

$$\sup\left\{ \sum_{w \in T_j^{(k)}} h[\mathbf{y}, \mathbf{p}](w) \;\middle|\; k \in \mathbb{N} \right\} = \sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_j} h[\mathbf{y}, \mathbf{p}](w) = \|\mathbf{y} = \mathbf{p}\|_j$$

for $j = 1, \ldots, t$. Summing everything up, we obtain

$$\sup\{\mathbf{p}^{(k)}(0, \ldots, 0) \mid k \in \mathbb{N}\} = (\|\mathbf{y} = \mathbf{p}\|_1, \ldots, \|\mathbf{y} = \mathbf{p}\|_t)^T,$$

and the proposition is proved. □

It is also obvious that our definition is consistent with the definition of weighted context-free grammars over *complete* semirings (these can be viewed as algebraic systems) as studied, e.g., by Droste and Vogler [6].[9]

Let us note that our approach is also consistent with the theory of *proper* algebraic systems over arbitrary power series semirings [17,19]. As mentioned in Section 2, such systems are known to have *unique proper solutions*, i.e., unique solutions $(r_1, \ldots, r_t)^T$ such that $(r_j, \varepsilon) = 0$ for $j = 1, \ldots, t$ [17,19]. Let S be a semiring and Σ be an alphabet. Let $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$ be a summation semiring such that $\mathcal{F}$ is the class of all locally finite families of power series in $\mathcal{F}(S\langle\langle\Sigma^*\rangle\rangle)$ and $\Phi(r_i \mid i \in I) = \sum_{i \in I} r_i = r$ for all $(r_i \mid i \in I)$ in $\mathcal{F}$ and r defined for all w in Σ^* by $(r, w) = \sum_{i \in I(w)} (r_i, w)$. Each algebraic system over S and Σ can then be viewed as an $S\langle\Sigma^*\rangle$-algebraic system over $(S\langle\langle\Sigma^*\rangle\rangle, \mathcal{F}, \Phi)$. Moreover, it is straightforward to prove that the canonical solution of each *proper* algebraic system viewed in this way is defined and proper. Hence, it is the unique proper solution. The detailed proof is left for the reader.

[9] Droste and Vogler [6] have in fact dealt with grammars over valuation monoids, thus going beyond semirings.

We shall now prove that if we take a summable family of homomorphic images of words from a language that is unambiguous context-free, but not necessarily a template of an algebraic system, then the sum over this family is still an algebraic element.

Lemma 2 *Let L be a language over a nonempty finite alphabet Σ, let $(S, \mathcal{F}, \Phi)$ be a summation semiring, and let $h\colon \Sigma^* \to (S, \cdot)$ be a monoid homomorphism. If the language L is unambiguous context-free and $(h(w) \mid w \in L)$ is in $\mathcal{F}$, then $\Phi(h(w) \mid w \in L) = \sum_{w \in L} h(w)$ is an S'-algebraic element of S, where S' is the set $S' = h(\Sigma) \cup \{0, 1\}$.*

Proof Let the language $L \subseteq \Sigma^*$ be generated by an unambiguous context-free grammar $\mathcal{G} = (N, \Sigma, P, \sigma)$.[10] Let $\mathcal{G}' = (N, \Gamma, P', \sigma)$ be a grammar constructed from $\mathcal{G}$ by labelling terminal symbols so that each occurs in precisely one production rule precisely once. This can be done for instance as follows: if $\pi = (\xi \to c_1 \dots c_k)$ is a production rule in P, where ξ is in N and $c_1, \dots, c_k$ are in $N \cup \Sigma$, then let $\varphi(\pi)$ be a production rule $\xi \to d_1 \dots d_k$, where for $i = 1, \dots, k$,

$$d_i = \begin{cases} [c_i, \pi, i] & \text{if } c_i \text{ is in } \Sigma, \\ c_i & \text{if } c_i \text{ is in } N. \end{cases}$$

Moreover, with the same notation as above, let

$$\Gamma(\pi) = \{d_i \mid i \in \{1, \dots, k\}, c_i \in \Sigma\}.$$

One may then take $\Gamma = \bigcup_{\pi \in P} \Gamma(\pi)$ (without loss of generality, assume that $N \cap \Gamma = \emptyset$) and $P' = \{\varphi(\pi) \mid \pi \in P\}$.

Let $g\colon \Gamma^* \to \Sigma^*$ be the label-removing homomorphism: $g([c, \pi, i]) = c$ for all $[c, \pi, i]$ in Γ. Next, let $\mathcal{G}'' = (N'', \Gamma \cup \Delta, P'', \sigma)$ be a context-free grammar constructed from $\mathcal{G}'$ by first inserting new terminals $\#_{\pi,0}, \dots, \#_{\pi,k}$ into the right hand side $d_1 \dots d_k$ of each production rule $\pi = (\xi \to d_1 \dots d_k)$ so that it starts and ends with a terminal and contains no consecutive nonterminals, next inserting a new nonterminal ζ between each two consecutive terminals on right hand sides of production rules, and finally adding the rule $\zeta \to c_\zeta$, where c_ζ is a new terminal. Formally, let $\Delta = \{c_\zeta\} \cup \bigcup_{\pi \in P'} \Delta(\pi)$, where $\Delta(\pi) = \{\#_{\pi,0}, \dots, \#_{\pi,k}\}$ for each $\pi = (\xi \to d_1 \dots d_k)$ in P' with ξ in N and $d_1, \dots, d_k$ in $\Gamma \cup N$, and assume that all symbols in Δ are new. Moreover, let $N'' = N \cup \{\zeta\}$, where ζ is a new symbol. Finally, for each $\pi = (\xi \to d_1 \dots d_k)$ in P', where ξ is in N and $d_1, \dots, d_k$ are in $\Gamma \cup N$, let

$$\psi(\pi) = (\xi \to \#_{\pi,0} z_1 d_1 z_1 \#_{\pi,1} z_2 d_2 z_2 \#_{\pi,2} z_3 d_3 \dots d_{k-1} z_{k-1} \#_{\pi,k-1} z_k d_k z_k \#_{\pi,k}),$$

where

$$z_i = \begin{cases} \zeta & \text{if } d_i \text{ is in } \Gamma, \\ \varepsilon & \text{if } d_i \text{ is in } N, \end{cases}$$

[10] Here, N stands for a nonempty finite alphabet of nonterminals, Σ stands for a nonempty finite alphabet of terminals, $N \cap \Sigma = \emptyset$, $P \subseteq N \times (N \cup T)^*$ is a set of production rules, and σ in N is the initial nonterminal.

for $i = 1, \ldots, k$. The set P'' is then given by

$$P'' = \{\psi(\pi) \mid \pi \in P'\} \cup \{\zeta \to c_\zeta\}.$$

Let $f\colon (\Gamma \cup \Delta)^* \to \Sigma^*$ be a homomorphism such that $f(c) = g(c)$ for each c in Γ and $f(c) = \varepsilon$ for each c in Δ. Then $\mathcal{G}''$ is an unambiguous grammar corresponding to a $\left(2_1^{\Gamma\cup\Delta} \cup \{\emptyset, \{\varepsilon\}\}\right)$-algebraic system over $2^{(\Gamma\cup\Delta)^*}$ (here, $2_1^{\Gamma\cup\Delta}$ denotes the set of all singleton subsets of $\Gamma \cup \Delta$), which is (after a suitable renaming of terminal symbols) a template system for an S'-algebraic system $\mathbf{y} = \mathbf{p}$ over the summation semiring S with $h[\mathbf{y}, \mathbf{p}] = h \circ f$. This in particular means that if $\|\mathcal{G}''\|$ is the language generated by $\mathcal{G}''$, then $h(f(x)) = h[\mathbf{y}, \mathbf{p}](x)$ for each x in $\|\mathcal{G}''\|$. Moreover, clearly

$$L = \bigcup_{x \in \|\mathcal{G}''\|} \{f(x)\};$$

as $\mathcal{G}$ is unambiguous, this union is obviously disjoint. It then follows by summability of $(h(w) \mid w \in L)$ and by conditions (i) and (ii) of Definition 2 that the family

$$(h(f(x)) \mid x \in \|\mathcal{G}''\|) = (h[\mathbf{y}, \mathbf{p}](x) \mid x \in \|\mathcal{G}''\|)$$

is summable and that

$$\sum_{w \in L} h(w) = \sum_{x \in \|\mathcal{G}''\|} h(f(x)) = \sum_{x \in \|\mathcal{G}''\|} h[\mathbf{y}, \mathbf{p}](x) = \|\mathbf{y} = \mathbf{p}\|_1.$$

As a result, $\sum_{w \in L} h(w)$ is an S'-algebraic element of S. □

5 Pushdown Automata over Summation Semirings

We shall now introduce pushdown automata over summation semirings and prove their equivalence with algebraic systems.

Definition 7 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring and S' a subset of S containing the elements 0 and 1. A *pushdown S'-automaton* over S is a sextuple $\mathcal{A} = (Q, \Gamma, T, q_0, Z_0, F)$, where $Q \neq \emptyset$ is a finite set of states, $\Gamma \neq \emptyset$ is a finite pushdown alphabet, $T \subseteq Q \times S' \times \Gamma \times Q \times \Gamma^*$ is a finite set of transitions, q_0 in Q is the initial state, Z_0 in Γ is the bottom-of-pushdown symbol, and $F \subseteq Q$ is a set of final states.

If (p, a, Z, q, γ) in T is a transition, then the automaton can make a step from a configuration with state p and a symbol Z on the top of the pushdown to a configuration with state q and with the upmost symbol of the pushdown replaced by the word γ (the rightmost symbol of γ being the new upmost symbol of the pushdown), while "reading" the semiring element a. With this interpretation, it should be clear how the language $\|\mathcal{A}\|$ recognised by $\mathcal{A}$ (by state) is defined in case $S = 2^{\Sigma^*}$ for some alphabet Σ and $S' = 2_1^{\Sigma} \cup \{\emptyset, \{\varepsilon\}\}$

(here, 2_1^Σ denotes the set of all singleton subsets of Σ). Similarly as for algebraic systems, we shall now use this special case to define the behaviour of pushdown automata over arbitrary summation semirings.[11]

Definition 8 Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and $\mathcal{A} = (Q, \Gamma, T, q_0, Z_0, F)$ a pushdown S'-automaton over S. Let us denote by 2_1^T the set of all singleton subsets of T. A *template automaton* corresponding to the automaton $\mathcal{A}$ is a pushdown $(2_1^T \cup \{\emptyset, \{\varepsilon\}\})$-automaton $\mathrm{temp}(\mathcal{A}) = (Q, \Gamma, T', q_0, Z_0, F)$ over 2^{T^*} such that

$$T' = \{(p, (p, a, Z, q, \gamma), Z, q, \gamma) \mid (p, a, Z, q, \gamma) \in T\}.$$

Let $\|\mathrm{temp}(\mathcal{A})\|$ be the language recognised by the template automaton $\mathrm{temp}(\mathcal{A})$ by state. Let $h[\mathcal{A}]\colon T^* \to (S, \cdot)$ be a monoid homomorphism given for each (p, a, Z, q, γ) in T by $h[\mathcal{A}](p, a, Z, q, \gamma) = a$. If $(h[\mathcal{A}](w) \mid w \in \|\mathrm{temp}(\mathcal{A})\|)$ is in $\mathcal{F}$, then we shall write

$$\|\mathcal{A}\| := \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w)$$

and call the semiring element $\|\mathcal{A}\|$ the *behaviour* of $\mathcal{A}$. Otherwise we shall say that the behaviour of $\mathcal{A}$ is undefined.

Remark 3 Note that the template automaton $\mathrm{temp}(\mathcal{A})$ is deterministic for each pushdown S'-automaton $\mathcal{A}$ over S. In particular, this implies that the language $\|\mathrm{temp}(\mathcal{A})\|$ is necessarily unambiguous context-free.

Let us now prove a natural counterpart to Lemma 2, which states that if we take a summable family of homomorphic images of words from a language that is unambiguous context-free, then the sum over this family is a behaviour of some pushdown automaton.

Lemma 3 *Let L be a language over a nonempty finite alphabet Σ, let $(S, \mathcal{F}, \Phi)$ be a summation semiring, and let $h\colon \Sigma^* \to (S, \cdot)$ be a monoid homomorphism. If L is unambiguous context-free and $(h(w) \mid w \in L)$ is in $\mathcal{F}$, then $\Phi(h(w) \mid w \in L) = \sum_{w \in L} h(w)$ is a behaviour of some pushdown S'-automaton over S, where $S' = h(\Sigma) \cup \{0, 1\}$.*

Proof Let us denote by 2_1^Σ the set of all singleton subsets of the alphabet Σ and L be recognised by an unambiguous $(2_1^\Sigma \cup \{\emptyset, \{\varepsilon\}\})$-pushdown automaton $\mathcal{A} = (Q, \Gamma, T, q_0, Z_0, F)$ over 2^{Σ^*} such that z is in $\Sigma \cup \{\varepsilon\}$ for each (p, z, Z, q, γ) in T. Without loss of generality, let us assume that if transitions (p, z, Z, q, γ) and (p, z', Z, q, γ) are both in T for some p, q in K, Z in Γ, γ in Γ^*, and z, z' in $\Sigma \cup \{\varepsilon\}$, then $z = z'$. Let $\mathcal{A}' = (Q, \Gamma, T', q_0, Z_0, F)$, where $T' = \{(p, (p, h(z), Z, q, \gamma), Z, q, \gamma) \mid (p, z, Z, q, \gamma) \in T\}$. Then $\mathcal{A}'$ is a template automaton for some pushdown S'-automaton $\mathcal{A}'' = (Q, \Gamma, T'', q_0, Z_0, F)$

[11] For the sake of correctness, let us note that the following definition is sound only because it is consistent with the usual definition for automata over semirings of formal languages. This is a hidden proposition, which is nevertheless easy to prove.

over S. Let $f\colon (T'')^* \to \Sigma^*$ be a homomorphism given for all (p, z, Z, q, γ) in T by $f(p, h(z), Z, q, \gamma) = z$ (this is a valid definition by our earlier assumption on transitions). Then $h[\mathcal{A}''] = h \circ f$. Moreover, unambiguity of $\mathcal{A}$ implies that L can be expressed by a *disjoint* union

$$L = \bigcup_{x \in \|\mathcal{A}'\|} \{f(x)\}.$$

Hence, it follows by summability of $(h(w) \mid w \in L)$ and by conditions (i) and (ii) of Definition 2 that the family

$$(h(f(x)) \mid x \in \|\mathcal{A}'\|) = (h[\mathcal{A}''](x) \mid x \in \|\mathcal{A}'\|)$$

is summable and

$$\sum_{w \in L} h(w) = \sum_{x \in \|\mathcal{A}'\|} h(f(x)) = \sum_{x \in \|\mathcal{A}'\|} h[\mathcal{A}''](x) = \|\mathcal{A}''\|,$$

which is the behaviour of the pushdown S'-automaton $\mathcal{A}''$ over S. □

We are now ready to prove the equivalence of pushdown automata over summation semirings with algebraic systems over summation semirings.

Theorem 1 *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and a in S. Then a is S'-algebraic over S if and only if $a = \|\mathcal{A}\|$ for some pushdown S'-automaton $\mathcal{A}$ over S with defined behaviour.*

Proof Let a be S'-algebraic over S and $\mathbf{y} = \mathbf{p}$ be an S'-algebraic system such that $a = \|\mathbf{y} = \mathbf{p}\|_1$. The language $\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1$ is unambiguous context-free and it follows by Lemma 3 that

$$a = \|\mathbf{y} = \mathbf{p}\|_1 = \sum_{w \in \|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1} h[\mathbf{y}, \mathbf{p}](w)$$

is a behaviour of some pushdown S'-automaton over S.

For the converse, let $a = \|\mathcal{A}\|$ for some pushdown S'-automaton $\mathcal{A}$ over S. Then $\|\mathrm{temp}(\mathcal{A})\|$ is unambiguous context-free and it follows by Lemma 2 that

$$a = \|\mathcal{A}\| = \sum_{w \in \|\mathrm{temp}(\mathcal{A})\|} h[\mathcal{A}](w)$$

is S'-algebraic over S, completing the proof. □

6 The Chomsky-Schützenberger Theorem

Let us finally generalise the Chomsky-Schützenberger theorem [2] to algebraic elements of summation semirings.

Theorem 2 *Let $(S, \mathcal{F}, \Phi)$ be a summation semiring, S' a subset of S containing 0 and 1, and a an S'-algebraic element of S. Then there exists a nonempty finite "left-bracket" alphabet Y, a monoid homomorphism $h\colon (Y \cup \overline{Y})^* \to (S, \cdot)$, and a rational language $R \subseteq (Y \cup \overline{Y})^*$ such that $(h(w) \mid w \in D_Y \cap R)$ is in $\mathcal{F}$ and*

$$a = \sum_{w \in D_Y \cap R} h(w),$$

where $\overline{Y} = \{\overline{y} \mid y \in Y\}$, $Y \cap \overline{Y} = \emptyset$, and D_Y is the Dyck language over $Y \cup \overline{Y}$.

Proof Let $\mathbf{y} = \mathbf{p}$ be an S'-algebraic system over S such that $a = \|\mathbf{y} = \mathbf{p}\|_1$. By the classical Chomsky-Schützenberger theorem for languages [2], there is an alphabet Y, a homomorphism $h'\colon (Y \cup \overline{Y})^* \to \Sigma[\mathbf{y}, \mathbf{p}]^*$, and a rational language $R \subseteq (Y \cup \overline{Y})^*$ such that $\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1 = h'(D_Y \cap R) = \bigcup_{w \in D_Y \cap R} \{h'(w)\}$. Moreover, it is not hard to see that since $\|\mathbf{y} = \mathrm{temp}(\mathbf{p})\|_1$ is unambiguous, the restriction of h' to $D_Y \cap R$ can be assumed to be injective. Taking $h = h[\mathbf{y}, \mathbf{p}] \circ h'$ and using conditions (i) and (ii) of Definition 2 thus completes the proof. □

References

1. Bloom, S. L., Ésik, Z., Kuich, W.: Partial Conway and Iteration Semirings. Fundam. Inform. 86, 19–40 (2008)
2. Chomsky, N., Schützenberger, M.-P.: The Algebraic Theory of Context-Free Languages. In: Braffort, P., Hirschberg, D. (eds.) Computer Programming and Formal Systems. pp. 118-161. North Holland, Amsterdam (1963)
3. Davey, B. A., Priestley, H. A.: Introduction to Lattices and Order 2nd ed. Cambridge University Press, Cambridge (2002)
4. Droste, M., Kuich, W., Vogler, H. (eds.) Handbook of Weighted Automata. Springer, Berlin Heidelberg (2009)
5. Droste, M., Kuich, W.: Semirings and Formal Power Series. In: Droste, M., Kuich, W., Vogler, H. (eds.) Handbook of Weighted Automata. Monographs in Theoretical Computer Science, pp. 3–28. Springer, Berlin Heidelberg (2009)
6. Droste, M., Vogler, H.: The Chomsky-Schützenberger Theorem for Quantitative Context-Free Languages. Int. J. Found. Comput. Sci. 25, 955–970 (2014)
7. Ésik, Z., Kuich, W.: A Unifying Kleene Theorem for Weighted Finite Automata. In: Calude, C. S., Rozenberg, G., Salomaa, A. (eds.) Maurer Festschrift. LNCS, vol. 6570, pp. 76–89. Springer, Berlin Heidelberg (2011)
8. Ésik, Z., Kuich, W.: Inductive *-Semirings. Theor. Comput. Sci. 324, 3–33 (2004)
9. Ésik, Z., Kuich, W.: Modern Automata Theory. http://www.dmg.tuwien.ac.at/kuich/mat.pdf
10. Hebisch, U., Weinert, H. J.: Semirings – Algebraic Theory and Applications in Computer Science. World Scientific, Singapore (1998)
11. Hopcroft, J. E., Ullman, J. D.: Introduction to Automata Theory, Languages, and Computation. Addison-Wesley, Reading (1979)
12. Korenjak, A. J., Hopcroft, J. E.: Simple Deterministic Languages. In: SWAT 1966. pp. 36–46. IEEE Computer Society, Washington, DC (1966)

13. Kostolányi, P., Rovan, B.: Automata with Auxiliary Weights. Int. J. Found. Comput. Sci. 27, 787–807 (2016)
14. Kuich, W.: Semirings and Formal Power Series: Their Relevance to Formal Languages and Automata. In: Rozenberg, G., Salomaa, A. (eds.) Handbook of Formal Languages, vol. 1. pp. 609–677. Springer, Berlin Heidelberg (1997)
15. Kuich, W., Salomaa, A.: Semirings, Automata, Languages. Springer, Berlin Heidelberg (1986)
16. Lausch, H., Nöbauer, W.: Algebra of Polynomials. North Holland, Amsterdam (1973)
17. Petre, I., Salomaa, A.: Algebraic Systems and Pushdown Automata. In: Droste, M., Kuich, W., Vogler, H. (eds.) Handbook of Weighted Automata. Monographs in Theoretical Computer Science, pp. 257–289. Springer, Berlin Heidelberg (2009)
18. Sakarovitch, J.: Elements of Automata Theory. Cambridge University Press, Cambridge (2009)
19. Salomaa, A., Soittola, M.: Automata-Theoretic Aspects of Formal Power Series. Springer, New York (1978)
20. Stanat, D. F.: A Homomorphism Theorem for Weighted Context-Free Grammars. J. Comput. Syst. Sci. 6, 217–232 (1972)