Polynomially Ambiguous Unary Weighted Automata over Fields

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Abstract

Every univariate rational series over an algebraically closed field is shown to be realised by some polynomially ambiguous unary weighted automaton. Unary weighted automata over algebraically closed fields thus always admit polynomially ambiguous equivalents. On the other hand, it is shown that this property does not hold over any other field of characteristic zero, generalising a recent observation about unary weighted automata over the field of rational numbers.

Keywords: Weighted Automaton, Formal Power Series, Difference Equation, Linear Recurrence, Polynomial Ambiguity, Field

1 Introduction

Weighted automata over fields and one-letter alphabets, which are the object of study in this article, are known to exhibit several special properties relating them to other important concepts in mathematics. First of all, such automata realise the usual univariate formal power series instead of languages, which has naturally led to the idea of viewing the objects realised by weighted automata over general alphabets as noncommutative formal power series [4, 8, 24]. Moreover, unary weighted finite automata over the complex numbers are known to realise precisely the formal Maclaurin expansions of rational functions analytic at $z = 0$ – for reasons like this one, the series realised by weighted finite automata are usually called rational.

The formalism of weighted regular expressions, equivalent to weighted finite automata, essentially boils down to the description of rational functions via polynomial fractions in the particular setting of unary alphabets and complex weights. This is the reason why regular expressions are often – and perhaps more meaningfully – called rational, even when they are used to describe languages [24]. The study of rational series and weighted automata, unary or not, has a long history dating back to the foundational article of M.-P. Schützenberger [26] – see, e.g., [4, 8, 9, 24].

Every univariate formal power series is uniquely described by its coefficient sequence – in fact, the difference between a series and its coefficient sequence is only a matter of interpretation. It is a classical observation that coefficient sequences of rational series over fields coincide with the sequences arising as solutions to initial value problems for linear difference equations – *i.e.*, linear recurrences – with constant coefficients. As a consequence, unary weighted automata over fields can also be understood as devices realising such sequences; see, for instance, C. Barloy et al. [1] or J. Bell and D. Smertnig [2].

The gist of the connection between unary weighted automata over fields and difference equations lies nevertheless in the fact that a unary weighted automaton can be naturally interpreted as an initial value problem for a linear autonomous system of first-order difference equations. By the classical theory of linear difference equations [10], such systems can be seen as "equivalent" to stand-alone higher-order linear difference equations with constant coefficients: given a system of k first-order difference equations, the characteristic polynomial of its matrix determines an "equivalent" k-th order difference equation; conversely, a k-th order difference equation always corresponds to a system whose $k \times k$ matrix is the companion matrix of the equation's characteristic polynomial.

We stick with a classical viewpoint in this article and understand unary weighted automata as devices realising formal power series. The connection to difference equations nevertheless remains crucial for our developments.

Degrees of ambiguity have initially been studied as a structural measure for nondeterministic finite automata without weights, especially from the viewpoint of descriptional complexity [22, 27]. Weighted automata with restricted ambiguity have later attracted significant attention as well. This research has often been motivated by the idea that algorithmic problems that cannot be easily solved for unrestricted weighted automata might become more tractable when the automaton is required to be, e.g., polynomially or finitely ambiguous. Such questions have been studied for weighted automata over tropical semirings [14–16] and for probabilistic automata [3]. The class of polynomially ambiguous weighted automata furthermore arises in connection to the weighted first-order logic of M. Droste and P. Gastin [6], and restricted ambiguity has also been considered for weighted tree automata [18, 21].

The most widely studied classes of weighted automata with restricted ambiguity are comprised by the *unambiguous*, the *finitely ambiguous*, the *polynomially ambiguous*, and the *unrestricted* weighted finite automata. This increasing chain of weighted automata classes has been referred to as the *ambiguity hierarchy* [1, 5, 18]. Although the inclusions in this hierarchy are strict on the level of automata, their strictness on the level of realised power series depends on the properties of the underlying semiring – for instance, all these classes of automata are equally powerful over finite semirings, as can be easily seen from the equivalence of deterministic and unrestricted weighted automata that holds in this case [17].

Examples separating certain levels of the ambiguity hierarchy for power series have been presented, e.g., by D. Kirsten [14] over the tropical semiring of integers, and by M. Droste and P. Gastin [6] over the semiring of natural numbers and over the tropical semirings. These results have gradually led to the observation that the ambiguity hierarchy is strict on the level of power series over the tropical semirings [5], and to the same observation over the field of rational numbers [1].

The most important part of the result of C. Barloy et al. [1], establishing strictness of the ambiguity hierarchy of power series over the rationals, separates the polynomially ambiguous and the unrestricted rational series over the rational numbers – it is shown there that the (univariate) generating function for the Fibonacci numbers, although clearly rational, is not realised by a polynomially ambiguous weighted automaton *over the rationals*. Similar results have been independently obtained by M. Droste and P. Gastin [6], and by F. Mazowiecki and C. Riveros [19] via a connection to cost-register automata. Nevertheless, the approach of C. Barloy et al. [1] is by far the most systematic: they prove a number of characterisations of polynomially ambiguous *unary* weighted automata over the rationals, e.g., via what they call poly-rational expressions and via eigenvalues of linear recurrences. The latter can be directly used to obtain the negative result for the Fibonacci numbers.

In the present article, we aim to deepen our understanding of polynomially ambiguous unary weighted finite automata over fields; the particular case when the underlying field is that of the rational numbers has already been explored by C. Barloy et al. [1]. We mostly rely on an elementary structural characterisation of polynomially ambiguous unary automata (Theorem 3.2) and on the spectral properties of their matrices.

We show that every univariate rational series over an algebraically closed field is realised by some polynomially ambiguous unary weighted automaton. This means that unary alphabets are not sufficient to separate the two upmost levels of the ambiguity hierarchy over algebraically closed fields.[1] Our observation is intimately linked to existence of the Jordan canonical form for matrices over algebraically closed fields.

For fields that are not algebraically closed, we identify a fundamental reason behind existence of univariate rational series that cannot be realised by a polynomially ambiguous unary weighted automaton. This in a sense explains why the example based on Fibonacci numbers works over the rationals, and – more

[1]We do not consider automata over other than unary alphabets in this article – we thus leave the relation between polynomially ambiguous and unrestricted weighted finite automata over algebraically closed fields open for general alphabets.

importantly – allows us to construct such examples over an arbitrary field of characteristic zero that is not algebraically closed. As a consequence, the inclusion between the two upmost levels of the ambiguity hierarchy is strict over such fields, and unary alphabets are sufficient to establish this property.

In other words, our main results can be summarised as follows: polynomially ambiguous unary weighted automata are expressive equivalents of unrestricted unary weighted automata over algebraically closed fields, and they are strictly less expressive than unrestricted unary weighted automata over fields of characteristic zero that are not algebraically closed.

2 Preliminaries

Fields are understood to be commutative. The symbol $\mathbb{N}$ is used to denote the set of all *nonnegative* integers, while $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote, respectively, the fields of rational, real, and complex numbers. Alphabets are always nonempty and finite; the empty word over any alphabet Σ is denoted by ε. For S a set and $m, n \in \mathbb{N}$, the set of all $m \times n$ matrices over S is denoted by $S^{m \times n}$; for S a semiring, the identity $n \times n$ matrix over S is denoted by $\mathbf{I}_n$.

Let us briefly collect some basic facts related to formal power series and weighted automata – the reader is encouraged to consult the chapters [7, 11, 25] of the handbook [8], as well as, e.g., [4, 9, 24] for a more comprehensive treatment.

A (noncommutative) *formal power series* over a semiring S and alphabet Σ is a mapping $r\colon \Sigma^* \to S$ interpreted in a slightly unusual way: the value of r upon $w \in \Sigma^*$ is denoted by (r, w) instead of $r(w)$ and called the *coefficient* of w in r; the series r itself is written as

$$r = \sum_{w \in \Sigma^*} (r, w)\, w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle \Sigma^* \rangle\rangle$.

The *sum* of series $r, s \in S\langle\langle \Sigma^* \rangle\rangle$ is a series $r + s$ defined for all $w \in \Sigma^*$ by

$$(r + s, w) = (r, w) + (s, w);$$

the *Cauchy product* of $r, s \in S\langle\langle \Sigma^* \rangle\rangle$ is a series $r \cdot s = rs$ given for all $w \in \Sigma^*$ by

$$(r \cdot s, w) = \sum_{\substack{u,v \in \Sigma^* \\ uv = w}} (r, u)(s, v).$$

Moreover, each $a \in S$ can be identified with a series $r_a \in S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_a, \varepsilon) = a$ and $(r_a, w) = 0$ for all $w \in \Sigma^+$; the algebra $(S\langle\langle \Sigma^* \rangle\rangle, +, \cdot, 0, 1)$ is then a semiring, again [7]. Similarly, each word $w \in \Sigma^*$ can be identified with a series $r_w \in S\langle\langle \Sigma^* \rangle\rangle$ such that $(r_w, w) = 1$ and $(r_w, x) = 0$ for all $x \in \Sigma^* \setminus \{w\}$. We may thus, e.g., write aw for the series – a so-called *monomial* – with coefficient a at w and all other coefficients zero.

When Σ is a unary alphabet – say, $\Sigma = \{c\}$ – and R is an integral domain, the semiring $R\langle\langle c^* \rangle\rangle$ coincides with the usual integral domain $R[\![c]\!]$ of formal power series in a single variable c with coefficients in R. This situation arises in particular when coefficients are taken from a field – that is, $\mathbb{F}\langle\langle c^* \rangle\rangle$ and $\mathbb{F}[\![c]\!]$ can be used interchangeably for a field $\mathbb{F}$.

A family $(r_i \mid i \in I)$ of series r_i from $S\langle\langle \Sigma^* \rangle\rangle$ – for S a semiring, Σ an alphabet, and I an arbitrary index set – is *locally finite* if the set $I(w) = \{i \in I \mid (r_i, w) \neq 0\}$ is finite for all $w \in \Sigma^*$. In that case, the *sum* over the family can be defined by

$$\sum_{i \in I} r_i = r,$$

where $r \in S\langle\langle \Sigma^* \rangle\rangle$ is a series whose coefficients are given, for each $w \in \Sigma^*$, by a *finite* sum

$$(r, w) = \sum_{i \in I(w)} (r_i, w).$$

A *weighted finite automaton* over a semiring S and alphabet Σ is a quadruple $\mathcal{A} = (Q, \sigma, \iota, \tau)$, where Q is a finite set of states, $\sigma\colon Q \times \Sigma \times Q \to S$ is a transition weighting function, $\iota\colon Q \to S$ is an initial weighting function, and $\tau\colon Q \to S$ is a terminal weighting function. A *run* of $\mathcal{A}$ is a word $\gamma = q_0 c_1 q_1 c_2 q_2 \ldots q_{n-1} c_n q_n \in (Q\Sigma)^* Q$ with $n \in \mathbb{N}$, $q_0, \ldots, q_n \in Q$, and $c_1, \ldots, c_n \in \Sigma$ such that $\sigma(q_{k-1}, c_k, q_k) \neq 0$ for $k = 1, \ldots, n$.

Consider a run $\gamma = q_0 c_1 q_1 c_2 q_2 \ldots q_{n-1} c_n q_n$ of the automaton $\mathcal{A}$ with $n \in \mathbb{N}$, $q_0, \ldots, q_n \in Q$, and $c_1, \ldots, c_n \in \Sigma$. We define the *label* of γ to be the word $\lambda(\gamma) = c_1 \ldots c_n$ and the *value* of γ to be the semiring element $\sigma(\gamma) = \sigma(q_0, c_1, q_1)\sigma(q_1, c_2, q_2) \ldots \sigma(q_{n-1}, c_n, q_n)$; we also say that γ is a run *upon* $\lambda(\gamma)$ *from* q_0 *to* q_n. We say that γ is *successful* if $\iota(q_0) \neq 0$ and $\tau(q_n) \neq 0$. The *monomial* $\|\gamma\| \in S\langle\langle \Sigma^* \rangle\rangle$ *realised by a run* γ from q_0 to q_n is defined by

$$\|\gamma\| = (\iota(q_0)\sigma(\gamma)\tau(q_n))\ \lambda(\gamma).$$

We denote the set of all runs of $\mathcal{A}$ by $\mathcal{R}(\mathcal{A})$ and the set of all successful runs of $\mathcal{A}$ by $\mathcal{R}_s(\mathcal{A})$. Moreover, given $w \in \Sigma^*$, let $\mathcal{R}(\mathcal{A}, w) = \{\gamma \in \mathcal{R}(\mathcal{A}) \mid \lambda(\gamma) = w\}$ and $\mathcal{R}_s(\mathcal{A}, w) = \{\gamma \in \mathcal{R}_s(\mathcal{A}) \mid \lambda(\gamma) = w\}$. The family $(\|\gamma\| \mid \gamma \in \mathcal{R}(\mathcal{A}))$ is clearly locally finite. We may thus define the *behaviour* of $\mathcal{A}$ by

$$\|\mathcal{A}\| = \sum_{\gamma \in \mathcal{R}(\mathcal{A})} \|\gamma\|.$$

It is easy to see that this can be equivalently expressed, again summing over a locally finite family of series, by

$$\|\mathcal{A}\| = \sum_{\gamma \in \mathcal{R}_s(\mathcal{A})} \|\gamma\|,$$

and that the coefficient at each $w \in \Sigma^*$ in $\|\mathcal{A}\|$ is given by

$$(\|\mathcal{A}\|, w) = \sum_{\gamma \in \mathcal{R}(\mathcal{A}, w)} (\|\gamma\|, w) = \sum_{\gamma \in \mathcal{R}_s(\mathcal{A}, w)} (\|\gamma\|, w)\,.$$

In particular, $\|\mathcal{A}\| = 0$ when $Q = \emptyset$. We also say that the series $\|\mathcal{A}\|$ is *realised* by the automaton $\mathcal{A}$. Series $r \in S\langle\langle \Sigma^* \rangle\rangle$ realised by weighted finite automata over S and Σ, or equivalently by *weighted rational expressions* over S and Σ [24], are called *rational* over S and Σ.

By a *weighted automaton*, we always understand a weighted *finite* automaton in what follows. Moreover, we often confine ourselves to state sets of the form $[n] = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. We then write $\mathcal{A} = (n, \sigma, \iota, \tau)$ as a shorthand for $\mathcal{A} = ([n], \sigma, \iota, \tau)$.

Given a weighted automaton $\mathcal{A}$ over S and Σ, define a function $\mathrm{amb}_{\mathcal{A}} \colon \Sigma^* \to \mathbb{N}$ to count the numbers of successful runs of $\mathcal{A}$ on words over Σ – that is, for all $w \in \Sigma^*$ by

$$\mathrm{amb}_{\mathcal{A}}(w) = |\mathcal{R}_s(\mathcal{A}, w)|\,.$$

We then say that $\mathcal{A}$ is *unambiguous* if $\mathrm{amb}_{\mathcal{A}}(w) \leq 1$ for all $w \in \Sigma^*$; *finitely ambiguous* if there exists $k \in \mathbb{N}$ such that $\mathrm{amb}_{\mathcal{A}}(w) \leq k$ for all $w \in \Sigma^*$; and *polynomially ambiguous* if there exists a polynomial function $p \colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{amb}_{\mathcal{A}}(w) \leq p(|w|)$ for all $w \in \Sigma^*$.

A weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over S and Σ is *accessible* if for each $q \in Q$ there exists a run from some $p \in Q$ such that $\iota(p) \neq 0$ to q; *coaccessible* if for each $p \in Q$ there exists a run from p to some $q \in Q$ such that $\tau(q) \neq 0$; and *trim* if it is both accessible and coaccessible.

Every weighted automaton over a semiring S and alphabet Σ can also be interpreted as a *linear S-representation* over Σ – *i.e.*, a quadruple $\mathcal{P} = (n, \mathbf{i}, \mu, \mathbf{f})$, where $n \in \mathbb{N}$ is the *order* of $\mathcal{P}$, $\mathbf{i} \in S^{1 \times n}$ is a vector of initial weights, $\mu \colon (\Sigma^*, \cdot, \varepsilon) \to (S^{n \times n}, \cdot, \mathbf{I}_n)$ is a monoid homomorphism, and $\mathbf{f} \in S^{n \times 1}$ is a vector of terminal weights. The *series realised by* $\mathcal{P}$ is defined by

$$\|\mathcal{P}\| = \sum_{w \in \Sigma^*} (\mathbf{i}\mu(w)\mathbf{f})\, w.$$

A series $r \in S\langle\langle \Sigma^* \rangle\rangle$ is *recognisable* if it is realised by a linear S-representation over the alphabet Σ.

Now, a weighted automaton $\mathcal{A} = (n, \sigma, \iota, \tau)$ over S and Σ can also be seen as a linear S-representation $\mathcal{P}_{\mathcal{A}}$ over Σ given by $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, \mu, \mathbf{f})$, where $\mathbf{i} = (\iota(1), \ldots, \iota(n))$, the matrix $\mu(c) = (a_{i,j})_{n \times n}$ is given for every $c \in \Sigma$ by $a_{i,j} = \sigma(i, c, j)$ for $i, j = 1, \ldots, n$, and $\mathbf{f} = (\tau(1), \ldots, \tau(n))^T$. It is easy to prove that $\|\mathcal{P}_{\mathcal{A}}\| = \|\mathcal{A}\|$. The correspondence $\mathcal{A} \mapsto \mathcal{P}_{\mathcal{A}}$ is moreover clearly bijective – every linear representation thus also corresponds to a unique weighted automaton. These observations establish the classical result, by which the sets of rational and recognisable series over S and Σ coincide [24, 25].

When $\Sigma = \{c\}$ is a unary alphabet and $\mathcal{P} = (n, \mathbf{i}, \mu, \mathbf{f})$ is a linear S-representation over Σ, the monoid homomorphism μ is uniquely determined by the matrix $A = \mu(c)$. We thus also write $\mathcal{P} = (n, \mathbf{i}, A, \mathbf{f})$ in this particular case, which is nevertheless the one that interests us the most.

Let $\mathcal{A} = (n, \sigma, \iota, \tau)$ be a weighted automaton over a *field* $\mathbb{F}$ and *unary* alphabet $\Sigma = \{c\}$ with $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$. Then it can also be interpreted as an initial value problem for the first-order linear autonomous system of difference equations (*i.e.*, recurrences)

$$\mathbf{x}_{t+1} = A\mathbf{x}_t \qquad \text{for all } t \in \mathbb{N} \tag{1}$$

with initial conditions given by $\mathbf{x}_0 = \mathbf{f}$. Indeed, the j-th component of the solution vector $\mathbf{x}_t$ of this system represents, for $j = 1, \ldots, n$ and all $t \in \mathbb{N}$, the coefficient at c^t in the series realised by an automaton obtained from $\mathcal{A}$ by setting $\iota(j) = 1$ and $\iota(k) = 0$ for all $k \in [n] \setminus \{j\}$. As a consequence, the coefficient at c^t in the behaviour of $\mathcal{A}$ is given, for each $t \in \mathbb{N}$, by

$$\left(\|\mathcal{A}\|, c^t\right) = \mathbf{i} \cdot \mathbf{x}_t.$$

The classical theory of difference equations [10] allows us to express the particular components of the solution vector $\mathbf{x}_t = (x_1(t), \ldots, x_n(t))^T$ of (1) in closed form over the *algebraic closure* $\overline{\mathbb{F}}$ of $\mathbb{F}$. Indeed, as a consequence of the similarity of A to a matrix over $\overline{\mathbb{F}}$ in Jordan canonical form,[2] it follows that for each $j \in [n]$, the function $x_j(t)$ can be written as

$$x_j(t) = \sum_{\lambda \in \sigma} \sum_{k=0}^{\alpha(\lambda)-1} b_{\lambda,k} \binom{t}{k} \lambda^{t-k} \qquad \text{for all } t \in \mathbb{N}, \tag{2}$$

where σ denotes the spectrum of A over $\overline{\mathbb{F}}$, the algebraic multiplicity of each eigenvalue λ of A is denoted by $\alpha(\lambda)$, and $b_{\lambda,k} \in \overline{\mathbb{F}}$ for all $\lambda \in \sigma$ and $k = 0, \ldots, \alpha(\lambda) - 1$.[3] This also implies that

$$\left(\|\mathcal{A}\|, c^t\right) = \sum_{\lambda \in \sigma} \sum_{k=0}^{\alpha(\lambda)-1} b'_{\lambda,k} \binom{t}{k} \lambda^{t-k} \qquad \text{for all } t \in \mathbb{N}, \tag{3}$$

where $b'_{\lambda,k}$, for all $\lambda \in \sigma$ and $k = 0, \ldots, \alpha(\lambda) - 1$, are constants from $\overline{\mathbb{F}}$. Recall that the eigenvalues of A are precisely the roots over $\overline{\mathbb{F}}$ of its *characteristic polynomial*

$$\mathrm{ch}_A(x) = \det\left(x\mathbf{I}_n - A\right),$$

[2] We get $\mathbf{x}_t = A^t\mathbf{f} = P^{-1}J^tP\mathbf{f}$ for some matrices $J, P \in \overline{\mathbb{F}}$ and all $t \in \mathbb{N}$, where J is in Jordan canonical form and P is invertible. Now, an easy combinatorial argument can be employed to observe that each entry of J^t takes the form $\binom{t}{k}\lambda^{t-k}$ for some eigenvalue λ of A over $\overline{\mathbb{F}}$ and $k \in \mathbb{N}$ that is smaller than the algebraic multiplicity of λ. The entries of P, P^{-1}, and $\mathbf{f}$ are constants (they do not depend on t). See also [10, Subsection 3.3.2].

[3] The binomial coefficient $\binom{t}{k}$ is a nonnegative integer, so it should be interpreted as a sum of $\binom{t}{k}$ ones in $\overline{\mathbb{F}}$.

the polynomial itself having coefficients in $\mathbb{F}$; the algebraic multiplicity of an eigenvalue $\lambda \in \overline{\mathbb{F}}$ is its multiplicity as a root of $\mathrm{ch}_A(x)$.

The constants $b_{\lambda,k}$ of (2) as well as $b'_{\lambda,k}$ of (3) are uniquely determined as solutions to systems of linear equations. Given functions $f_1, \ldots, f_n \colon \mathbb{N} \to \overline{\mathbb{F}}$ and $t \in \mathbb{N}$, the *Casorati matrix*[4] $\mathrm{Cas}(t)$ of $f_1, \ldots, f_n$ is a matrix

$$\mathrm{Cas}(t) = \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1(t+1) & f_2(t+1) & \cdots & f_n(t+1) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(t+n-1) & f_2(t+n-1) & \cdots & f_n(t+n-1) \end{pmatrix}.$$

The Casorati matrix $\mathrm{Cas}(0)$ of the functions $\binom{t}{k}\lambda^{t-k}$ for $\lambda \in \sigma$ and $k = 0, \ldots, \alpha(\lambda) - 1$ is a *generalised Vandermonde matrix* [10, 13], which is known to be nonsingular. As a result, after calculating the values $x_j(0), \ldots, x_j(n-1)$ using (1), the constants $b_{\lambda,k}$ can be determined as a unique solution to a system of n linear equations given by (2) for $t = 0, \ldots, n-1$, and similarly for the constants $b'_{\lambda,k}$. This also leads to the following observation.

Proposition 2.1 *Let $\mathbb{F}$ be a field. Any set of pairwise distinct functions $f \colon \mathbb{N} \to \overline{\mathbb{F}}$ of the form $f(t) = \binom{t}{k}\lambda^{t-k}$ with $\lambda \in \overline{\mathbb{F}}$ and $k \in \mathbb{N}$ is then linearly independent.*

3 Polynomially Ambiguous Unary Automata

The ambiguity degree of a weighted automaton of course does not depend on its (nonzero) weights – each weighted automaton $\mathcal{A}$ over an arbitrary semiring S and alphabet Σ is thus polynomially ambiguous if and only if this property holds for a nondeterministic finite automaton without weights obtained from $\mathcal{A}$ by simply "forgetting about the weights".[5]

Decidable structural characterisations of polynomially ambiguous finite automata are well known [4, 22, 27]. Let us recall the criterion described by A. Weber and H. Seidl [27].

Theorem 3.1 (A. Weber and H. Seidl [27]) *Let $\mathcal{A}$ be a trim finite automaton over an alphabet Σ with state set Q. Then $\mathcal{A}$ is polynomially ambiguous if and only if there is no $q \in Q$ with at least two distinct runs from q to q upon the same $w \in \Sigma^*$.*

This characterisation admits a particularly simple form for unary automata, which we record in Theorem 3.2 below. By what has been said above, we may state this result for unary *weighted* automata over an arbitrary semiring.

[4] The determinant of the Casorati matrix is usually called the *Casoratian* and is a discrete counterpart of the *Wrońskian*, which is important for the theory of linear *differential* equations.

[5] More precisely, one only distinguishes between zero and nonzero weights and interprets this distinction over the Boolean semiring.

The graph of a weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over a semiring S and unary alphabet $\Sigma = \{c\}$ is understood in a usual way – its vertices are states of $\mathcal{A}$ and edges correspond to transitions of $\mathcal{A}$, *i.e.*, they are given by pairs $(p, q) \in Q^2$ such that $\sigma(p, c, q) \neq 0$.

Theorem 3.2 *Let S be a semiring and $\mathcal{A}$ a trim unary weighted automaton over S and $\Sigma = \{c\}$. Then $\mathcal{A}$ is polynomially ambiguous if and only if the strongly connected components of its graph are all either single vertices, or directed cycles.*

Proof First, let $\mathcal{A}$ be polynomially ambiguous and suppose that its graph contains a strongly connected component other than a single vertex or a directed cycle. Then $\mathcal{A}$ contains a state q, for which there are two distinct runs γ_1, γ_2 from q to q such that $\lambda(\gamma_1) = c^s$ and $\lambda(\gamma_2) = c^t$ for some $s, t \in \mathbb{N} \setminus \{0\}$, and both runs begin by a different transition. By repeating t times the run γ_1 and s times the run γ_2, we obtain two distinct runs from q to q upon c^{st}, contradicting Theorem 3.1.

Conversely, let the strongly connected components of the graph of $\mathcal{A}$ be all either single vertices, or directed cycles. Then it is clear that given a state q and $t \in \mathbb{N}$, there can be at most one run upon c^t from q to q. The automaton $\mathcal{A}$ is thus polynomially ambiguous by Theorem 3.1. □

Alternatively, one can use the Perron-Frobenius theory (see, e.g., [20]) to establish Theorem 3.2. Let S be a semiring and $n \in \mathbb{N} \setminus \{0\}$, and first consider a trim unary weighted automaton $\mathcal{A}$ over S and $\Sigma = \{c\}$ with a *strongly connected* graph and with $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$. The number of successful runs of $\mathcal{A}$ upon c^t can be expressed, for all $t \in \mathbb{N}$, by $\mathrm{amb}_{\mathcal{A}}(c^t) = \nu(\mathbf{i})\nu(A)^t\nu(\mathbf{f})$, where $\nu\colon S \to \mathbb{N}$ is a mapping applied componentwise and defined for all $a \in S$ by

$$\nu(a) = \begin{cases} 1 \text{ if } a \neq 0, \\ 0 \text{ if } a = 0. \end{cases}$$

Let ϱ be the Perron-Frobenius eigenvalue of $\nu(A)$, *i.e.*, an eigenvalue equal to the spectral radius of $\nu(A)$. By Perron-Frobenius theory, there exists a number $p \in [n]$, called the *period* of the graph of $\mathcal{A}$ or the *imprimitivity index* of $\nu(A)$ [20, Chapter 3], such that for $k = 0, \ldots, p-1$, one either has $\mathrm{amb}_{\mathcal{A}}(c^t) = \Theta(\varrho^t)$ for $t \to \infty$ with $t \equiv k \pmod{p}$, or $\mathrm{amb}_{\mathcal{A}}(c^t) = 0$ for all $t \in \mathbb{N}$ with $t \equiv k \pmod{p}$, while there is at least one k for which the first possibility occurs. Now, $\varrho = 0$ whenever $\mathcal{A}$ consists of one state without a loop and $\varrho = 1$ whenever the graph of $\mathcal{A}$ is a directed cycle through all states. In these cases, $\mathcal{A}$ is polynomially ambiguous. When $\mathcal{A}$ is not of this form, its matrix $\nu(A)$ "strictly" dominates a matrix with spectral radius 1 (e.g., an adjacency matrix of a factor of the original graph of $\mathcal{A}$, whose edges form a directed cycle, not necessarily through all states). It thus follows by Wielandt's theorem [20, Theorem 2.1] that $\varrho > 1$, so that $\mathcal{A}$ is not polynomially ambiguous.

Note also that it follows from what has been said above that a polynomially ambiguous *strongly connected* unary weighted automaton is actually always finitely ambiguous.[6]

Now, consider the general case of otherwise unrestricted *trim* unary weighted automata. Let $\mathcal{A}$ with $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$ be such automaton over a semiring S and one-letter alphabet $\Sigma = \{c\}$. For $\nu\colon S \to \mathbb{N}$ defined as above, consider the matrix $\nu(A) \in \mathbb{N}^{n\times n}$ and the associated initial value problem for the first-order linear autonomous system of difference equations

$$\mathbf{x}_{t+1} = \nu(A)\mathbf{x}_t \qquad \text{for all } t \in \mathbb{N}$$

with initial conditions given by $\mathbf{x}_0 = \nu(\mathbf{f})$. Then for all $t \in \mathbb{N}$, the value $\mathrm{amb}_{\mathcal{A}}(c^t)$ is obviously given by $\nu(\mathbf{i}) \cdot \mathbf{x}_t$. The components of $\mathbf{x}_t$ take the form of (2).[7] Now, the spectrum of $\nu(A)$ is the multiset union of spectra of matrices corresponding to particular strongly connected components. This means that when all strongly connected components of the graph of $\mathcal{A}$ are either single vertices or directed cycles, all eigenvalues of $\nu(A)$ are of modulus 1 or 0. The exponential factors in (2) vanish as a consequence, implying that $\mathrm{amb}_{\mathcal{A}}(c^t)$ can be bounded from above by a polynomial function in t – $\mathcal{A}$ is polynomially ambiguous. On the other hand, if the graph of $\mathcal{A}$ contains a strongly connected component of some other type and one chooses some state q from this component, it follows by what has been said above that the number of runs γ from q to q with $\lambda(\gamma) = c^t$ cannot be bounded from above by a polynomial in t. As $\mathcal{A}$ is trim, the same property holds for the number of successful runs upon c^t in $\mathcal{A}$, *i.e.*, for $\mathrm{amb}_{\mathcal{A}}(c^t)$ – the automaton $\mathcal{A}$ is not polynomially ambiguous.

Finally, note that the structural characterisation of Theorem 3.2 can be further refined – when $\mathcal{A}$ is polynomially ambiguous, the lowest degree of a polynomial function, by which $\mathrm{amb}_{\mathcal{A}}(c^t)$ can be bounded from above, equals the highest possible number of distinct directed cycles that a single run of $\mathcal{A}$ can pass through. Once again, this is a particular case of a known characterisation for automata over general alphabets [27].

4 The Case of Algebraically Closed Fields

We now examine unary weighted automata over *algebraically closed* fields. In sharp contrast with the recent result of C. Barloy et al. [1] for unary automata over the field of rational numbers, we show that *every* unary weighted automaton over an algebraically closed field $\mathbb{F}$ is equivalent to some *polynomially ambiguous* unary weighted automaton over $\mathbb{F}$.

[6] The only possible source of ambiguity in such automata is related to the fact that an automaton may have multiple states with nonzero initial weights.

[7] One may view this system over $\mathbb{N}$ as a system over the field of rational or complex numbers as well.

Theorem 4.1 *Let $\mathbb{F}$ be an algebraically closed field, and $\mathcal{A}$ a unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$. Then there is a polynomially ambiguous unary weighted automaton $\mathcal{J}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\|\mathcal{J}\| = \|\mathcal{A}\|$.*

Proof Let $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$. As the field $\mathbb{F}$ is algebraically closed, there exists a matrix $J \in \mathbb{F}^{n \times n}$ in Jordan canonical form and an invertible matrix $P \in \mathbb{F}^{n \times n}$ such that $A = P^{-1}JP$. Let us construct a unary weighted automaton $\mathcal{J}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\mathcal{P}_{\mathcal{J}} = (n, \mathbf{i}P^{-1}, J, P\mathbf{f})$. Then $\|\mathcal{J}\| = \|\mathcal{A}\|$, as

$$\left(\|\mathcal{J}\|, c^t\right) = \mathbf{i}P^{-1}J^tP\mathbf{f} = \mathbf{i}P^{-1}\left(PAP^{-1}\right)^t P\mathbf{f} = \mathbf{i}A^t\mathbf{f} = \left(\|\mathcal{A}\|, c^t\right)$$

for all $t \in \mathbb{N}$.[8] Moreover, $\mathcal{J}$ is polynomially ambiguous by Theorem 3.2, as the matrix J in Jordan canonical form is necessarily upper triangular, implying that all strongly connected components of the graph of $\mathcal{J}$ consist of a single vertex.[9] □

Let us now briefly describe an alternative proof of Theorem 4.1, in which the polynomially ambiguous equivalent of $\mathcal{A}$ is constructed from the closed form (3) for the coefficient sequence of $\|\mathcal{A}\|$; note that $\overline{\mathbb{F}} = \mathbb{F}$ when $\mathbb{F}$ is algebraically closed, implying that the eigenvalues and constants in (3) belong directly to $\mathbb{F}$ in our case.

First, observe that for any semiring S, $\lambda \in S$, $h \in \mathbb{N}$, and $b_0, \ldots, b_h \in S$, there is a polynomially ambiguous unary weighted automaton $\mathcal{J}_{\lambda,h}[b_0, \ldots, b_h]$ over S and $\Sigma = \{c\}$ such that for all $t \in \mathbb{N}$,

$$\left(\|\mathcal{J}_{\lambda,h}[b_0, \ldots, b_h]\|, c^t\right) = \sum_{k=0}^{h} b_k \binom{t}{k} \lambda^{t-k}; \tag{4}$$

λ^{t-k} is undefined for $t < k$, but we assume by convention that $\binom{t}{k}\lambda^{t-k} = 0$ in that case.

Indeed, one can take $\mathcal{J}_{\lambda,h}[b_0, \ldots, b_h] = (Q, \sigma, \iota, \tau)$ with $Q = \{0, \ldots, h\}$,

$$\sigma(p, c, q) = \begin{cases} \lambda \text{ if } q = p, \\ 1 \text{ if } q = p - 1, \\ 0 \text{ otherwise} \end{cases}$$

for all $p, q \in Q$, $\iota(k) = b_k$ for $k = 0, \ldots, h$, $\tau(0) = 1$, and $\tau(k) = 0$ for $k = 1, \ldots, h$. For $h = 3$, the automaton just constructed is depicted in Fig. 1.

This automaton is polynomially ambiguous by Theorem 3.2, and (4) follows by observing that for $k = 0, \ldots, h$ and each $t \in \mathbb{N}$, there are exactly $\binom{t}{k}$ runs

[8] In fact, we could just note that $\mathcal{A}$ and $\mathcal{J}$ are evidently *similar* [24]. The observation established can be rephrased as a well-known fact that similar automata are always equivalent.

[9] Although the automaton $\mathcal{J}$ does not have to be trim in general, it can be turned into a trim automaton by possibly removing several states. It is clear that the nature of strongly connected components is not spoiled by this process, and that any of the two automata is polynomially ambiguous if and only if the other automaton is. Theorem 3.2 thus can be invoked.

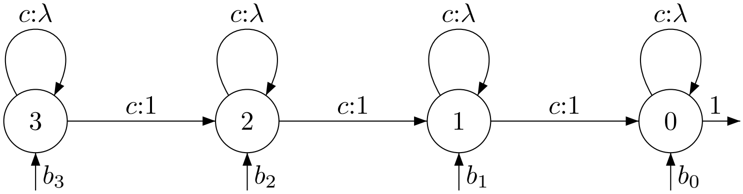

Figure 1: The automaton $\mathcal{J}_{\lambda,3}[b_0, b_1, b_2, b_3]$ for $\lambda, b_0, \ldots, b_3 \in S$.

upon c^t from k to 0, while one has $t \geq k$ and $\|\gamma\| = (b_k \lambda^{t-k})\, c^t$ for every such run, and no other run upon c^t leading from k can be successful.

It now remains to make use of the standard binary operation of *union* on automata: given weighted automata $\mathcal{A}_1$, $\mathcal{A}_2$ over a semiring S and alphabet Σ, the state set of $\mathcal{A}_1 \cup \mathcal{A}_2$ is a disjoint union of state sets of both automata, while the transitions of $\mathcal{A}_1 \cup \mathcal{A}_2$ are defined as in $\mathcal{A}_1$ between states from $\mathcal{A}_1$ and as in $\mathcal{A}_2$ between states from $\mathcal{A}_2$; there are no transitions joining states from $\mathcal{A}_1$ with states from $\mathcal{A}_2$, and the initial and terminal weights of states are also inherited from the original automata. One thus merely "places $\mathcal{A}_1$ besides $\mathcal{A}_2$", so that

$$\|\mathcal{A}_1 \cup \mathcal{A}_2\| = \|\mathcal{A}_1\| + \|\mathcal{A}_2\|.$$

It is clear that $\mathcal{A}_1 \cup \mathcal{A}_2$ is polynomially ambiguous whenever $\mathcal{A}_1$ and $\mathcal{A}_2$ are.

Given a unary weighted automaton $\mathcal{A}$ over an algebraically closed field $\mathbb{F}$ and $\Sigma = \{c\}$, it follows by (3) that there is a finite set $\varsigma \subseteq \mathbb{F}$, a mapping $h\colon \varsigma \to \mathbb{N}$, and constants $b_{\lambda,k} \in \mathbb{F}$ for $\lambda \in \varsigma$ and $k = 0, \ldots, h(\lambda)$ such that for all $t \in \mathbb{N}$,

$$(\|\mathcal{A}\|, c^t) = \sum_{\lambda \in \varsigma} \sum_{k=0}^{h(\lambda)} b_{\lambda,k} \binom{t}{k} \lambda^{t-k},$$

i.e.,

$$\|\mathcal{A}\| = \sum_{\lambda \in \varsigma} \left\| \mathcal{J}_{\lambda,h(\lambda)}[b_{\lambda,0}, \ldots, b_{\lambda,h(\lambda)}] \right\|.$$

The automaton $\mathcal{A}$ is thus equivalent to a polynomially ambiguous automaton $\mathcal{J}'$ given by

$$\mathcal{J}' = \bigcup_{\lambda \in \varsigma} \mathcal{J}_{\lambda,h(\lambda)}[b_{\lambda,0}, \ldots, b_{\lambda,h(\lambda)}].$$

Note that although the construction of a polynomially ambiguous automaton just described is closely related to the one given in the proof of Theorem 4.1 and although the matrix J' of $\mathcal{P}_{\mathcal{J}'} = (n', \mathbf{i}', J', \mathbf{f}')$ is in the Jordan canonical form as well, the resulting automata are not the same in general.

The automaton $\mathcal{J}$ constructed in the proof of Theorem 4.1 is always similar to the original automaton $\mathcal{A}$, and thus it has the same number of states as $\mathcal{A}$. On the other hand, the latter construction allows to choose ς and h such that $b_{\lambda,h(\lambda)} \neq 0$ for all $\lambda \in \varsigma$. In that case, the resulting automaton $\mathcal{J}'$ is always a *minimal* automaton for the series $\|\mathcal{A}\|$.

Example 4.2 The unary weighted automaton $\mathcal{F}$ over $\mathbb{Q}$ and $\Sigma = \{z\}$ in Fig. 2 realises the generating function

$$F(z) = \sum_{t \in \mathbb{N}} F_t\, z^t = \frac{z}{1 - z - z^2}$$

of the Fibonacci numbers. That is, the sequence $(F_t)_{t=0}^{\infty}$ is given by $F_0 = 0$, $F_1 = 1$, and

$$F_{t+2} = F_{t+1} + F_t \qquad \text{for all } t \in \mathbb{N}.$$

It has been shown by C. Barloy et al. [1] that the series $F(z) \in \mathbb{Q}[\![z]\!]$ is not realised by any polynomially ambiguous unary weighted automaton over $\mathbb{Q}$.[10]

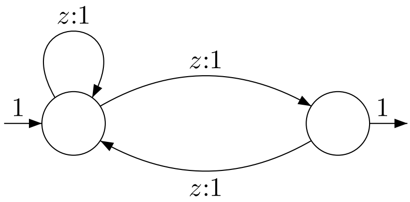

Figure 2: The automaton $\mathcal{F}$ realising the series $F(z)$.

On the contrary, it follows by Theorem 4.1 that the series $F(z)$ *is* realised by a polynomially ambiguous unary weighted automaton over the field $\overline{\mathbb{Q}}$ of algebraic numbers – or over the field $\mathbb{C}$ of complex numbers – and alphabet $\Sigma = \{z\}$. As

$$F_t = \frac{1}{\sqrt{5}}\varphi^t - \frac{1}{\sqrt{5}}\psi^t$$

with

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \psi = \frac{1-\sqrt{5}}{2}$$

for all $t \in \mathbb{N}$, the automaton $\mathcal{J}'$ equivalent to $\mathcal{F}$ is given by the diagram in Fig. 3.

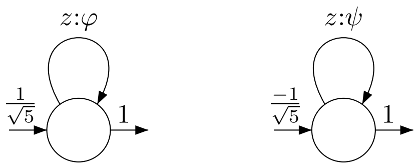

Figure 3: The automaton $\mathcal{J}'$ equivalent to $\mathcal{F}$.

In fact, $\mathcal{J}'$ can also be viewed as an automaton over the real algebraic numbers, but this is a mere coincidence – it follows by our later results that there is an univariate rational series over $\mathbb{Q}$ that cannot be realised by any polynomially ambiguous unary weighted automaton over the real algebraic numbers.

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[10] Note that the automaton $\mathcal{F}$ in Fig. 2 is not polynomially ambiguous by Theorem 3.2, as its graph consists of a single strongly connected component with two vertices, which does not take the form of a directed cycle.

5 Fields that Are Not Algebraically Closed

We now turn our attention to automata over fields that are not algebraically closed, for which the situation turns out to be radically different than for automata over algebraically closed fields. As already mentioned, C. Barloy et al. [1] have recently shown that polynomially ambiguous unary weighted automata over the field of rational numbers $\mathbb{Q}$ are strictly less powerful than unrestricted unary weighted automata over the same field. We now generalise this result by showing that the same holds for *every field of characteristic zero* that is not algebraically closed.

In order to establish this result, let us first identify an inherent reason for non-existence of polynomially ambiguous equivalents of unary weighted automata over general fields, which takes the form of the following sufficient condition. In what follows, $\mathbb{F}[x]$ denotes the ring of polynomials in one variable x with coefficients in $\mathbb{F}$.

Theorem 5.1 *Let $\mathbb{F}$ be a field over which there exists an* irreducible *polynomial $p(x) \in \mathbb{F}[x]$ that does not divide a polynomial of the form $x^n - a$ with $n \in \mathbb{N} \setminus \{0\}$ and $a \in \mathbb{F}$. Then there is a rational series r over $\mathbb{F}$ and $\Sigma = \{c\}$ that cannot be realised by a polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$.*

Proof Let $\mathbb{F}$ and $p(x) \in \mathbb{F}[x]$ be as in the statement of the theorem. Without loss of generality, let us suppose that $p(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ is monic. Consider a unary weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\mathcal{P}_{\mathcal{A}} = (m, \mathbf{i}, A, \mathbf{f})$, where A is the *companion matrix* of $p(x)$, *i.e.*,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{m-1} \end{pmatrix},$$

so that the characteristic polynomial of A equals $p(x)$. The automaton $\mathcal{A}$ is obviously strongly connected, as irreducibility of $p(x)$ implies that $a_0 \neq 0$. Moreover, let $\mathbf{i} = (1, 0, \ldots, 0)$ and $\mathbf{f}$ contain at least one nonzero component. Thus $\|\mathcal{A}\| \neq 0$.

Suppose for contradiction that there is a polynomially ambiguous unary weighted automaton $\mathcal{A}'$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\mathcal{P}_{\mathcal{A}'} = (m', \mathbf{i}', A', \mathbf{f}')$ and $\|\mathcal{A}'\| = \|\mathcal{A}\|$, and assume that $\mathcal{A}'$ is trim. As characteristic polynomials of matrices corresponding to directed cycles are always of the form $x^n - a$ for some $n \in \mathbb{N} \setminus \{0\}$ and $a \in \mathbb{F} \setminus \{0\}$, it follows by Theorem 3.2 that the characteristic polynomial of A' can be written as

$$\mathrm{ch}_{A'}(x) = x^{\ell} \prod_{j=1}^{s} \left(x^{n_j} - a_j\right) \tag{5}$$

for some $\ell, s \in \mathbb{N}$, $n_1, \ldots, n_s \in \mathbb{N} \setminus \{0\}$, and $a_1, \ldots, a_s \in \mathbb{F} \setminus \{0\}$.

Now, by linear independence of pairwise distinct functions of the form $\binom{t}{k}\lambda^{t-k}$ (Proposition 2.1), it follows that the expressions (3) for $(\|\mathcal{A}\|, c^t)$ and for $(\|\mathcal{A}'\|, c^t)$ have to be the same, which together with $\|\mathcal{A}\| \neq 0$ implies that A and A' have at least one eigenvalue $\lambda \in \overline{\mathbb{F}}$ in common. This eigenvalue λ thus has to be a root of $p(x)$ over $\overline{\mathbb{F}}$, so that $p(x) \in \mathbb{F}[x]$ is, by its irreducibility over $\mathbb{F}$, the minimal polynomial of λ with respect to the extension $\overline{\mathbb{F}}$ of $\mathbb{F}$. As λ is at the same time a root of $\mathrm{ch}_{A'}(x)$, the polynomial $p(x)$ has to divide $\mathrm{ch}_{A'}(x)$. By (5) and uniqueness of polynomial factorisation, this means that $p(x)$ divides either x or $x^{n_j} - a_j$ for some $j \in [s]$: a contradiction.

The series $r = \|\mathcal{A}\|$, rational over $\mathbb{F}$ and $\Sigma = \{c\}$, thus cannot be realised by a polynomially ambiguous weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$. □

Next, let us observe that the sufficient condition of Theorem 5.1 admits a simple equivalent form.

Proposition 5.2 *Let* $\mathbb{F}$ *be a field. An* irreducible *polynomial* $p(x) \in \mathbb{F}[x]$ *divides a polynomial of the form* $x^n - a$ *with* $n \in \mathbb{N} \setminus \{0\}$ *and* $a \in \mathbb{F}$ *if and only if for every two roots* ν, ξ *of* $p(x)$ *from the algebraic closure* $\overline{\mathbb{F}}$ *of* $\mathbb{F}$*, there exists* $n \in \mathbb{N} \setminus \{0\}$ *such that* $\nu^n = \xi^n \in \mathbb{F}$.

Proof The "only if" direction is clear. For the converse, assume that for every pair of roots $\nu, \xi \in \overline{\mathbb{F}}$ of $p(x)$, there exists $n(\nu, \xi) \in \mathbb{N} \setminus \{0\}$ such that $\nu^{n(\nu,\xi)} = \xi^{n(\nu,\xi)} \in \mathbb{F}$. Let n be the least common multiple of all such $n(\nu, \xi)$, so that λ^n has the same value in $\mathbb{F}$ for all roots $\lambda \in \overline{\mathbb{F}}$ of $p(x)$. Denote this common value by a. It follows that $p(x)$ divides the polynomial $(x^n - a)^k$ for some $k \in \mathbb{N} \setminus \{0\}$. Then, by irreducibility of $p(x)$ and uniqueness of polynomial factorisation, we conclude that $p(x)$ actually divides the polynomial $x^n - a$. □

Corollary 5.3 *Let* $\mathbb{F}$ *be a field, for which there exists an* irreducible *polynomial* $p(x) \in \mathbb{F}[x]$ *and roots* $\nu, \xi \in \overline{\mathbb{F}}$ *of* $p(x)$ *such that there is no* $n \in \mathbb{N} \setminus \{0\}$ *satisfying* $\nu^n = \xi^n \in \mathbb{F}$*. Then there is a rational series* r *over* $\mathbb{F}$ *and* $\Sigma = \{c\}$ *that cannot be realised by a polynomially ambiguous unary weighted automaton over* $\mathbb{F}$ *and* $\Sigma = \{c\}$.

We now show that the assumptions of Theorem 5.1 – or equivalently, of Corollary 5.3 – are satisfied by any field $\mathbb{F}$ *of characteristic zero* that is not algebraically closed. Let us start with the case when $\mathbb{F}$ is *uncountable.*

Lemma 5.4 *Let* $\mathbb{F}$ *be an uncountably infinite field of characteristic zero that is not algebraically closed. Then there exists a rational series* r *over* $\mathbb{F}$ *and* $\Sigma = \{c\}$ *that cannot be realised by a polynomially ambiguous unary weighted automaton over* $\mathbb{F}$ *and* $\Sigma = \{c\}$.

Proof As $\mathbb{F}$ is not algebraically closed, there exists a polynomial $p(x) \in \mathbb{F}[x]$ of degree at least two, irreducible over $\mathbb{F}$. Let $\nu, \xi \in \overline{\mathbb{F}}$ be two distinct roots of $p(x)$ over $\overline{\mathbb{F}}$.[11]

[11] Their existence follows from separability of irreducible polynomials over fields of characteristic zero (see, e.g., S. Roman [23, Corollary 1.6.3]).

We first show that there exists $a \in \mathbb{F} \setminus \{0\}$ such that $1 + a\nu$ and $1 + a\xi$ have no common positive integer power – that is, there is no $n \in \mathbb{N} \setminus \{0\}$ for which $(1 + a\nu)^n = (1 + a\xi)^n$. Indeed, every $b \in \overline{\mathbb{F}}$ such that $(1 + b\nu)^n = (1 + b\xi)^n$ for $n \in \mathbb{N} \setminus \{0\}$ is clearly a root of the polynomial

$$\Psi_n(x) = (1 + x\nu)^n - (1 + x\xi)^n$$

from $\overline{\mathbb{F}}[x]$. The coefficient at x in $\Psi_n(x)$ is $n(\nu - \xi)$. As the characteristic of $\overline{\mathbb{F}}$ has to be zero and as $\nu \neq \xi$, necessarily $n(\nu - \xi) \neq 0$. Thus $\Psi_n(x)$ is a nonzero polynomial of degree at most n – and as such, it has at most n distinct roots over $\overline{\mathbb{F}}$. As a consequence, we see that the set

$$B = \{b \in \overline{\mathbb{F}} \mid \exists n \in \mathbb{N} \setminus \{0\} : \Psi_n(b) = 0\}$$

is at most countably infinite. However, the field $\mathbb{F}$ is uncountably infinite, which means that $\mathbb{F} \setminus B$ is nonempty. Any $a \in \mathbb{F} \setminus B$ therefore has the desired property, *i.e.*, there is no $n \in \mathbb{N} \setminus \{0\}$ for which $(1 + a\nu)^n = (1 + a\xi)^n$, while obviously $a \neq 0$. Let such a be fixed for the rest of this proof.

Let us finally consider the polynomial $\overline{p}(x) = p\left(\frac{x-1}{a}\right) \in \mathbb{F}[x]$. Clearly, both $1 + a\nu$ and $1 + a\xi$ are roots of $\overline{p}(x)$. Moreover, $\overline{p}(x)$ is irreducible over $\mathbb{F}$, as otherwise we would have

$$\overline{p}(x) = \overline{p}_1(x) \cdot \overline{p}_2(x)$$

for some non-constant polynomials $\overline{p}_1(x), \overline{p}_2(x) \in \mathbb{F}[x]$, which would imply

$$p(x) = \overline{p}\,(ax + 1) = \overline{p}_1\,(ax + 1) \cdot \overline{p}_2\,(ax + 1) = p_1(x) \cdot p_2(x)$$

for non-constant polynomials $p_1(x) = \overline{p}_1\,(ax + 1)$ and $p_2(x) = \overline{p}_2\,(ax + 1)$ in $\mathbb{F}[x]$, contradicting irreducibility of $p(x)$. Existence of a rational series r from the statement of the lemma thus follows by Corollary 5.3. □

Let us now establish the same result for *countably* infinite fields of characteristic zero that are not algebraically closed.

Lemma 5.5 *Let $\mathbb{F}$ be a countably infinite field of characteristic zero that is not algebraically closed. Then there exists a rational series r over $\mathbb{F}$ and $\Sigma = \{c\}$ that cannot be realised by a polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$.*

Proof As $\mathbb{F}$ is of characteristic zero, its prime subfield $\mathbb{P}$ is isomorphic to $\mathbb{Q}$, and as $\mathbb{F}$ is countably infinite, its transcendence degree κ over $\mathbb{P}$ is surely at most $\aleph_0$. Let S with $|S| = \kappa$ be a subset of $\mathbb{R}$ that is algebraically independent over $\mathbb{Q}$, and consider the subfield $\mathbb{Q}(S)$ of $\mathbb{R}$. Then $\overline{\mathbb{F}}$ and $\overline{\mathbb{Q}(S)}$ are algebraically closed fields of the same transcendence degree over their isomorphic prime subfields – the fields $\overline{\mathbb{F}}$ and $\overline{\mathbb{Q}(S)}$ are thus isomorphic as well (see, e.g., T. W. Hungerford [12, Theorem VI.1.12]).

Let $\varphi\colon \overline{\mathbb{F}} \to \overline{\mathbb{Q}(S)}$ be an isomorphism. As $\mathbb{F}$ is not algebraically closed, it is isomorphic to a proper subfield $\mathbb{K} = \varphi(\mathbb{F})$ of $\overline{\mathbb{Q}(S)} \subseteq \mathbb{C}$, while $\overline{\mathbb{K}} = \overline{\mathbb{Q}(S)}$. We show existence of an irreducible polynomial $p(x) \in \mathbb{K}[x]$ and roots $\nu, \xi \in \overline{\mathbb{K}}$ of $p(x)$ such that there is no $n \in \mathbb{N} \setminus \{0\}$ satisfying $\nu^n = \xi^n \in \mathbb{K}$. Via the isomorphism φ, this clearly implies that the sufficient condition of Corollary 5.3 is fulfilled both for $\mathbb{K}$ and for $\mathbb{F}$, so that existence of the series r from the statement of the lemma is assured as well.

First, note that as $\overline{\mathbb{K}} = \overline{\mathbb{Q}(S)}$ with $S \subseteq \mathbb{R}$, the field $\overline{\mathbb{K}}$ contains precisely the roots of polynomials with coefficients from $\mathbb{Q}(S) \subseteq \mathbb{R}$; by the complex conjugate root theorem, this implies that $\overline{\mathbb{K}}$ is closed under complex conjugation.

Let $\alpha = a + bi$ with $a, b \in \mathbb{R}$ be an element of $\overline{\mathbb{K}} \setminus \mathbb{K}$. Then actually $a, b \in \mathbb{R} \cap \overline{\mathbb{K}}$, as closure of $\overline{\mathbb{K}}$ under complex conjugation together with $i \in \overline{\mathbb{Q}} \subseteq \overline{\mathbb{K}}$ implies that

$$a = \frac{\alpha + \overline{\alpha}}{2} \qquad \text{and} \qquad b = \frac{\alpha - \overline{\alpha}}{2i},$$

are both in $\overline{\mathbb{K}}$. If a, b, and i were all in $\mathbb{K}$, then α would be in $\mathbb{K}$ as well, contradicting our assumption. As a result, $\mathbb{K}$ either does not contain i, or it does not contain some *real* number $\eta \in \overline{\mathbb{K}}$.

If $\mathbb{K}$ does not contain the imaginary unit i, take

$$p(x) = x^2 - \frac{6}{5}x + 1$$

with roots

$$\nu = \frac{3 + 4i}{5} \qquad \text{and} \qquad \xi = \frac{3 - 4i}{5}.$$

Then $p(x) \in \mathbb{Q}[x] \subseteq \mathbb{K}[x]$ is irreducible over $\mathbb{K}$, as i would clearly be in $\mathbb{K}$ whenever ν or ξ was. Moreover, as any positive integer power of ν or ξ is of the form $a + bi$ for $a, b \in \mathbb{Q}$ – *i.e.*, it is a *Gaussian rational* – the imaginary unit i would be in $\mathbb{K}$ whenever any such number with $b \neq 0$ was in $\mathbb{K}$. Thus, if ν and ξ have a common positive integer power from $\mathbb{K}$, it is necessarily real. As $|\nu| = |\xi| = 1$, we can actually assume that

$$\nu^n = \xi^n = 1$$

for some $n \in \mathbb{N} \setminus \{0\}$. However, this is impossible, as ν and ξ are well known not to be complex roots of unity.[12] There is thus no $n \in \mathbb{N} \setminus \{0\}$ such that $\nu^n = \xi^n \in \mathbb{K}$, which finishes the proof in case $\mathbb{K}$ does not contain i.

For the remaining case, let us suppose that $\mathbb{K}$ does not contain some *real* number $\eta \in \overline{\mathbb{K}}$. Let $q(x) \in \mathbb{K}[x]$ be the minimal – hence irreducible – polynomial of η with respect to the extension $\overline{\mathbb{K}}$ of $\mathbb{K}$. If there exists a root ϑ of $q(x)$ such that $|\vartheta| \neq |\eta|$, then there clearly is no $n \in \mathbb{N} \setminus \{0\}$ such that $\vartheta^n = \eta^n$, and we may directly take $p(x) = q(x)$, $\nu = \eta$, and $\xi = \vartheta$. Otherwise, let us first note that the roots of $q(x)$ are all distinct by separability of irreducible polynomials over fields of characteristic zero. Take $p(x) = q(x - 1)$. This polynomial is irreducible and has $1 + \eta$ as a root. Moreover, it easily follows by distinctness of the roots of $q(x)$ that there is no other root of $p(x)$ with the same absolute value as $1 + \eta$. We may thus take η for ν and any other root of $p(x)$ for ξ – then surely $\nu^n \neq \xi^n$ for all $n \in \mathbb{N} \setminus \{0\}$. The lemma is proved in the remaining case as well. □

We may now summarise our findings by the following theorem, which is in a sense the main result of this section.

Theorem 5.6 *Let $\mathbb{F}$ be a field of characteristic zero that is not algebraically closed. Then there exists a rational series r over $\mathbb{F}$ and $\Sigma = \{c\}$ that cannot be realised by a polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$.*

Proof As every field of characteristic zero is infinite, the theorem follows directly by Lemma 5.5 and Lemma 5.4. □

[12] For instance, as $(3 + 4i)^2 = -7 + 24i \equiv 3 + 4i \pmod 5$, it follows that there is no $n \in \mathbb{N} \setminus \{0\}$ such that the real part of $(3 + 4i)^n$ equals 5^n, which has to hold whenever $\nu^n = 1$.

6 Conclusions

We have studied the power of polynomially ambiguous unary weighted automata over fields – more precisely, the question of whether they are expressive equivalents of unrestricted unary weighted automata. This question has already been answered in the particular case of automata over the rational numbers by C. Barloy et al. [1], who have shown that there are univariate rational series over $\mathbb{Q}$ that cannot be realised by a polynomially ambiguous automaton. This implies that the inclusion between the two upmost levels of the ambiguity hierarchy is strict for rational series over $\mathbb{Q}$.

In the two main results of this article, we have resolved this question for two classes of abstract fields. First, we have proved that polynomially ambiguous and unrestricted unary weighted automata are equivalent over algebraically closed fields. This result has thus most notably been established for the fields of complex and algebraic numbers. On the other hand, we have observed that among fields of characteristic zero, algebraically closed fields are the only fields with the said property. In other words, polynomially ambiguous unary weighted automata are strictly less powerful than unrestricted unary weighted automata over fields of characteristic zero that are not algebraically closed. This generalises the result from [1] and establishes the same observation, e.g., for automata over the real numbers or over algebraic number fields. We leave open the case of fields with positive characteristic.

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