# Riešenia tretej sady domácich úloh

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13. apríla 2016

**Úloha 1.** Vypočítajte:

a) $\sum_{k\in\mathbb{N}} 3^k \binom{n}{k}\binom{n}{3}$.

b) $\sum_{k\in\mathbb{N}} 3^k \binom{n}{k}\binom{k}{3}$.

c) $\sum_{k\in\mathbb{N}} 3^k \binom{n}{3k}$.

*Riešenie.*

a) Z binomickej vety dostávame

$$\sum_{k\in\mathbb{N}} 3^k \binom{n}{k}\binom{n}{3} = \binom{n}{3}\sum_{k\in\mathbb{N}} 3^k\binom{n}{k} = \binom{n}{3}(1+3)^n = \binom{n}{3}4^n.$$

b) Položme

$$F_n(x) := \sum_{k\in\mathbb{N}} x^k\binom{n}{k} = (1+x)^n =: G(x).$$

Potom

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}F_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} x^k\binom{n}{k} = \sum_{k=0}^{n}\frac{\mathrm{d}}{\mathrm{d}x}x^k\binom{n}{k} = \sum_{k=0}^{n} kx^{k-1}\binom{n}{k},\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} kx^{k-1}\binom{n}{k} = \sum_{k=0}^{n}\frac{\mathrm{d}}{\mathrm{d}x}kx^{k-1}\binom{n}{k} = \sum_{k=0}^{n} k(k-1)x^{k-2}\binom{n}{k},\\
\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n} k(k-1)x^{k-2}\binom{n}{k} = \sum_{k=0}^{n}\frac{\mathrm{d}}{\mathrm{d}x}k(k-1)x^{k-2}\binom{n}{k} =\\
&= \sum_{k=0}^{n} k(k-1)(k-2)x^{k-3}\binom{n}{k}
\end{aligned}$$

a

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}G_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(1+x)^n = n(1+x)^{n-1},\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}G_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}n(1+x)^{n-1} = n(n-1)(1+x)^{n-2},\\
\frac{\mathrm{d}^3}{\mathrm{d}x^3}G_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}n(n-1)(1+x)^{n-2} = n(n-1)(n-2)(1+x)^{n-3}.
\end{aligned}$$

Z toho dostávame

$$\begin{aligned}
\sum_{k\in\mathbb{N}} 3^k\binom{n}{k}\binom{k}{3} &= \frac{1}{6}\sum_{k\in\mathbb{N}} k(k-1)(k-2)3^k\binom{n}{k} = \frac{9}{2}\,\frac{\mathrm{d}^3}{\mathrm{d}x^3}F_n(x)\Big|_{x=3} =\\
&= \frac{9}{2}\,\frac{\mathrm{d}^3}{\mathrm{d}x^3}G_n(x)\Big|_{x=3} = \frac{9}{2}n(n-1)(n-2)4^{n-3}.
\end{aligned}$$

c) Zjavne

$$\sum_{k\in\mathbb{N}} 3^k \binom{n}{3k} = \frac{1}{3}\left(\sum_{k\in\mathbb{N}} 3^{k/3}\binom{n}{k} + \sum_{k\in\mathbb{N}} 3^{k/3}e^{2\pi ik/3}\binom{n}{k} + \sum_{k\in\mathbb{N}} 3^{k/3}e^{4\pi ik/3}\binom{n}{k}\right).$$

Z binomickej vety (rozšírenej do komplexného oboru) teda dostávame

$$\sum_{k\in\mathbb{N}} 3^k \binom{n}{3k} = \frac{1}{3}\left((1+\sqrt[3]{3})^n + (1+\sqrt[3]{3}e^{2\pi i/3})^n + (1+\sqrt[3]{3}e^{4\pi i/3})^n\right). \qquad \square$$