Riešenia tretej sady domácich úloh

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Úloha 1. Vypočítajte:

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{3k}.$$

Riešenie. Zjavne

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{3k} = \frac{1}{3}\left(\sum_{k\in\mathbb{N}} 2^k \binom{2n}{k} + \sum_{k\in\mathbb{N}} 2^k e^{2\pi i k/3}\binom{2n}{k} + \sum_{k\in\mathbb{N}} 2^k e^{4\pi i k/3}\binom{2n}{k}\right).$$

Z binomickej vety (rozšírenej do komplexného oboru) teda dostávame

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{3k} = \frac{1}{3}\left(3^{2n} + \left(1+2e^{2\pi i/3}\right)^{2n} + \left(1+2e^{4\pi i/3}\right)^{2n}\right). \qquad \square$$

Pokračovanie riešenia. Aj keď stačilo prísť s výsledkom uvedeným vyššie, v skutočnosti sa v tejto úlohe dá komplexných čísel zbaviť. Platí totiž

$$e^{2\pi i/3} = \frac{\sqrt{3}i - 1}{2} \qquad \text{a} \qquad e^{4\pi i/3} = \frac{-\sqrt{3}i - 1}{2}.$$

Z toho

$$1 + 2e^{2\pi i/3} = \sqrt{3}i \qquad \text{a} \qquad 1 + 2e^{4\pi i/3} = -\sqrt{3}i,$$

a teda

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{3k} = \frac{1}{3}\left(3^{2n} + (\sqrt{3}i)^{2n} + (-\sqrt{3}i)^{2n}\right) = \frac{1}{3}\left(9^n + 2(-1)^n 3^n\right). \qquad \square$$

Úloha 2. Vypočítajte:

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{k}\binom{2k}{2}.$$

Dôkaz. Položme

$$F_n(x) := \sum_{k\in\mathbb{N}} x^k \binom{2n}{k} = (1+x)^{2n} =: G(x).$$

Potom

$$\frac{\mathrm{d}}{\mathrm{d}x}F_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{2n} x^k \binom{2n}{k} = \sum_{k=0}^{2n} \frac{\mathrm{d}}{\mathrm{d}x} x^k \binom{2n}{k} = \sum_{k=1}^{2n} k x^{k-1} \binom{2n}{k},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=1}^{2n} k x^{k-1} \binom{2n}{k} = \sum_{k=1}^{2n} \frac{\mathrm{d}}{\mathrm{d}x} k x^{k-1} \binom{2n}{k} = \sum_{k=2}^{2n} k(k-1) x^{k-2} \binom{2n}{k}$$

a

$$\frac{\mathrm{d}}{\mathrm{d}x}G_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1+x)^{2n} = 2n(1+x)^{2n-1},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}G_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}2n(1+x)^{2n-1} = 2n(2n-1)(1+x)^{2n-2}$$

(ak $x = -1$ a $n = 0$, interpretujeme výrazy $2n(1+x)^{2n-1}$ a $2n(2n-1)(1+x)^{2n-2}$ ako nulu). Z toho dostávame

$$\sum_{k\in\mathbb{N}} 8^k \binom{2n}{k}\binom{2k}{2} = \sum_{k\in\mathbb{N}} 8^k(2k^2-k)\binom{2n}{k} = 2\sum_{k\in\mathbb{N}} 8^k k(k-1)\binom{2n}{k} + \sum_{k\in\mathbb{N}} 8^k k\binom{2n}{k} =$$
$$= 128\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_n(x)\bigg|_{x=8} + 8\,\frac{\mathrm{d}}{\mathrm{d}x}F_n(x)\bigg|_{x=8} = 128\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}G_n(x)\bigg|_{x=8} + 8\,\frac{\mathrm{d}}{\mathrm{d}x}G_n(x)\bigg|_{x=8} =$$
$$= 256n(2n-1)9^{2n-2} + 16n9^{2n-1} = (512n-112)n81^{n-1}. \qquad \square$$