

Genus of the cartesian product of n triangles.

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Abstract

We investigate the orientable genus of G_n , the cartesian product of n triangles, with a particular attention paid to the two smallest unsolved cases $n = 4$ and 5 . Using a lifting method we present a general construction of a low-genus embedding of G_n using a low-genus embedding of G_{n-1} . Combining this method with a computer search and a careful analysis of face structure we show that $30 \leq \gamma(G_4) \leq 37$ and $133 \leq \gamma(G_5) \leq 190$. Moreover, our computer search resulted in more than 1300 non-isomorphic minimum-genus embeddings of G_3 . We also introduce genus range of a group and (strong) symmetric genus range of a Cayley graph and of a group. The (strong) symmetric genus range of irredundant Cayley graphs of Z_p^n is calculated for all odd primes p .

Keywords: graph, cartesian product, genus, embedding, triangle, symmetric embedding, Cayley graph, Cayley map, genus range, group.

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1 Introduction

Finding the minimum genus of a graph is a very difficult problem both from practical and algorithmic perspective. Although in general it is NP-hard to determine the minimum genus [23] even in the class of cubic graphs [24], the minimum genus of various specific families of graphs has been determined in the past. A well-known instance is the determination of the minimum genus of complete graphs [22], which indicates the level of difficulty of determining the minimum genus. The genera of hypercubes have been computed by G. Ringel [21] and by Beineke and Harary [1]. Generalizations of these methods were used by A. T. White to calculate the genus of the cartesian products of even cycles [25] and later by T. Pisanski to cartesian products of more general graph classes [15, 16]. These techniques have been used to determine the genus of most abelian and hamiltonian groups and non-orientable genus of some metacyclic groups [17, 18, 19]. In most cases the developed methods can be adopted to the products where some factors are odd cycles of length at least five. The determination of genus of cartesian products involving triangular factors resisted all attempts in most cases, with the notable exceptions being the Cayley

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graphs of $Z_3 \times Z_3 \times Z_3$ [13, 5] and semi-direct product $Z_3 \rtimes Z_9$ [4]. The Cayley graphs of these groups are perhaps the most intriguing being the last groups of order at most 32 whose genus was determined. Moreover, if an abelian group Γ does not contain a Z_3 factor and its rank is at least 4, then the smallest genus of a Cayley graph for Γ follows from a formula in [10]. Therefore, it is not very surprising that the determination of the smallest genus of a Cayley graph for an abelian group containing a Z_3 factor is considered to be extremely difficult, see [9, Chapter 6] or [2, Chapter 11]. The aim of this paper is to determine good lower and upper bounds on the minimum among genera of Cayley graphs for Z_3^n with a particular attention paid to the smallest unsolved cases $n = 4$ and 5 . Our main result is the following theorem. It can be easily seen that every generating set of Z_3^n contains at least n elements and each Cayley graph of Z_3^n generated by precisely n elements is isomorphic to G_n , the cartesian product of n triangles.

Theorem 1.1. *The lower bound $L(n)$ and the upper bound $U(n)$ for the genus of G_n are given by Table 1. In particular, $30 \leq G_4 \leq 37$ and $130 \leq G_5 \leq 190$.*

n	1	2	3	4	5	$n \geq 5$
$L(n)$	0	1	7	30	133	$1 + \lceil 3^{n-1}(5n-12)/8 \rceil$
$U(n)$	0	1	7	37	190	$1 + 3^{n-2}(3n-8)$

Table 1: Lower and upper bounds on the genus of G_n .

Furthermore, motivated by results on genera of Cayley maps we introduce (symmetric) genus-range type invariants for Cayley graphs and groups which in a natural way broaden the concepts of symmetric genus of a Cayley graph and minimum genus of a group.

We assume that the reader is familiar with basics of topological graph theory as covered by, for instance, chapters 2 and 3 of [9] or chapters 5, 6, and 10 of [26]. In particular, we assume that the reader is familiar with (regular) voltage graphs. We use standard terminology consistent with [9] and consider only cellular embeddings into orientable surfaces.

The paper is organized as follows. We treat lower bounds in Section 2 and upper bounds in Section 3. Section 3 is additionally divided into three parts, the first discussing genus-range type parameters of groups and Cayley graphs, the second dealing with the computational aspects of the problem, and the third presenting a recursive construction of a low-genus embedding of G_n using a low-genus embedding of G_{n-1} .

2 Lower bounds

By G_n we denote the cartesian product of n triangles, that is,

$$G_n = \underbrace{K_3 \square K_3 \square \cdots \square K_3}_{n\text{-times}}.$$

For a prime p , by Z_p^n we denote the direct product of n copies of the cyclic group Z_p ; clearly, G_n is a Cayley graph of Z_3^n . For the number of vertices and edges of G_n we have $|V(G_n)| = 3^n$ and $|E(G_n)| = n3^n$, respectively. The total number of triangles of G_n is denoted by $T(G_n)$; clearly, $T(G_n) = n3^{n-1}$. For an embedding of G_n , the number of faces with length i of the embedding is denoted by f_i . For instance, f_3 is the number of triangular faces of the embedding. Faces of length three are called triangles, faces of length four are called rectangles, or quadrangular, and faces of length five are called pentagons. For an embedding Π of G_n , by F_Π we denote the number of faces of Π . An easy counting shows that if G_n would have an embedding with all faces being triangles, then the number of faces would be $2n3^{n-1}$, implying $\gamma(G_n) \geq 1 + \lceil 3^{n-1}(n-3)/2 \rceil$. For $n = 3$ and 4 this inequality implies $\gamma(G_3) \geq 1$ and $\gamma(G_4) \geq 15$. However, every edge of G_n lies in

precisely one triangle. In the union of face boundaries, every edge is traversed twice, once in every direction. Therefore, the maximum number of faces cannot be larger than if the second occurrence of every edge is in a face of length 4. An embedding of G_n such that every triangle bounds a face and every other face is quadrangular is called a *triangle-quadrangular* embedding. The genus of such embedding of G_n would be $1 + 3^{n-1}(5n - 12)/8$ and it is an open problem whether such embeddings exist for $n > 1$. Clearly, a triangle-quadrangular embedding may exist only if n is congruent to 4 (mod 8). This discussion can be summarized by the following proposition and two open problems.

Proposition 2.1. *The maximum number of faces in an embedding of G_n is at most $\lfloor |E(G_n)|/3 + |E(G_n)|/4 \rfloor$. Consequently $\gamma(G_n) \geq 1 + \lceil 3^{n-1}(5n - 12)/8 \rceil$, with equality holding if and only if G_n has a triangle-quadrangular embedding. \square*

For $n = 3, 4$, and 5, Proposition 2.1 gives $\gamma(G_3) \geq 5$, $\gamma(G_4) \geq 28$, and $\gamma(G_5) \geq 133$. Currently, Proposition 2.1 gives the best known lower bound on the genus of G_n for any $n > 4$.

Problem 1. *Is there an integer $n > 1$ such that G_n has a triangle-quadrangular embedding?*

Problem 2. *Determine all integers n such that G_n has a triangle-quadrangular embedding.*

Of a certain interest might be also quadrangular embeddings of G_n , that is, embeddings in which every face has length 4. Note that the well-known genus embedding of G_2 in the torus is quadrangular.

Problem 3. *Is there an integer $n > 2$ such that G_n has a quadrangular embedding?*

Our main result concerning lower bounds is Theorem 2.7 below, which asserts that $\gamma(G_4) \geq 30$. The proof is based on a method used by Brin and Squier in [5] to prove that $\gamma(G_3) \geq 6$. Note that the analysis used in [5] to prove that $\gamma(G_3) \geq 7$ is quite involved. Nevertheless, its use may lead to a better lower bound on the genus of G_4 . We start with necessary definitions. A *plane* is a subgraph of G_n obtained from G_n by fixing all but two coordinates. Clearly, a plane is isomorphic to G_2 and therefore every plane contains 9 vertices and 18 edges. It is easy to see that any face of length 3 or 4 in any embedding of G_n lies in some plane of G_n . A cycle of G is called *present* if it bounds a face, otherwise it is called *absent*. For a fixed embedding of G_n , let a denote the total number of absent triangles, that is $a = T(G_n) - f_3$. Let a_i denote the number of planes with precisely i absent triangles. The following two results can be proved by an easy counting.

Proposition 2.2. *Every triangle of G_n is lying in $n - 1$ planes of G_n . In particular, $(n - 1)a = \sum_{i=0}^6 ia_i$. \square*

Proposition 2.3. *The graph G_n contains precisely $\binom{n}{2} \cdot 3^{n-2}$ planes. \square*

The following result is Proposition 3 in [5].

Lemma 2.4. *If a plane of an embedding of G_n has i absent triangles, then it has at most m_i present rectangles, where the values of m_i are in the Table 2. \square*

i	0	1	2	3	4	5	6
m_i	1	2	2	3	4	5	6

Table 2: The values of m_i .

Proposition 2.5. *The graph G_4 does not have a triangle-quadrangular embedding.*

Proof. First note that every rectangle lies in precisely one plane of G_n . Assume that Π is a triangle-quadrangular embedding of G_4 . As Π has all triangles present, by Lemma 2.4 every plane has at most one present rectangle. By Proposition 2.3 the embedding contains at most 54 rectangles, which is a contradiction. \square

A calculation analogous to the proof of Proposition 2.5 does not exclude the possible existence of a triangle-quadrangular embedding of G_n for $n \geq 5$ such that $n \equiv 4 \pmod{8}$.

It was observed in [5, Proposition 2] that minimum-genus embeddings of G_n with the maximum number of present triangles do not contain faces of length 5. The next proposition is a slight extension of this result.

Proposition 2.6. *For any integer g such that $g \geq \gamma(G_n)$ there is an embedding of G_n in an orientable surface of genus at most g without faces of length 5.*

Proof. Let $g' = \min\{g, \gamma_M(G)\}$ and let Π be any embedding of G_n in the orientable surface of genus g' ; by Interpolation theorem for orientable surfaces (see [8]) and the choice of g' such an embedding exists. If $f_5 = 0$, then there is nothing to prove. Suppose that $f_5 > 0$ and that F is a face of length 5. It is easy to see that any pentagon has the form $aabab^{-1}$ for some generators of G_n a and b ; denote by v the vertex of F incident with two occurrences of a . It follows that Π does not contain the triangle T of the form aaa incident with v . Let e be the edge of T not contained in F . Moving e into the interior of F splits F into a triangular face bounded by T and a rectangle; denote the resulting embedding by Π' . Since the move of e can be replaced by removing e from Π and then adding it back in a different position and a removal of an edge changes the number of faces by at most one, the genus of Π' is not larger than the genus of Π . If the genera of Π and Π' are equal, then e lies on the boundary of two distinct faces of Π . The removal of e merges these two faces into a pentagon in Π' if and only if one of them is triangle and the other is rectangle in Π . Observe that e lies in precisely one triangle, the triangle T , which is absent in Π . Therefore, e does not lie in a face of length 3 in Π and the removal of e cannot merge two faces into a pentagon. It follows that Π' has either a smaller genus or a smaller number of pentagons than Π and repeating the process yields the desired embedding of G_n . \square

Theorem 2.7. *The genus of G_4 is at least 30.*

Proof. Let Π be an embedding of G_n . Since the union of face boundaries includes every edge precisely twice and every face that is not triangular has length at least four, we have

$$648 = 2|E(G_4)| = \sum_{i=0}^{\infty} i f_i \geq 3f_3 + 4(F_{\Pi} - f_4). \quad (1)$$

Let χ denote the Euler characteristic of the underlying surface. Euler formula implies $F_{\Pi} = \chi + |E(G_4)| - |V(G_4)|$. Substituting this equality into (1) and manipulating we get

$$\begin{aligned} 648 &\geq 3f_3 + 4(F_{\Pi} - f_4) \\ f_3 &\geq 4F_{\Pi} - 648 = 4\chi + 4|E(G_n)| - 4|V(G_n)| - 648 \\ &= 4\chi + 4 \cdot 324 - 4 \cdot 81 - 648 = 4\chi + 324. \end{aligned}$$

Consequently,

$$f_3 \geq 4\chi + 324. \quad (2)$$

By Proposition 2.6 we can assume that $f_5 = 0$. It follows that

$$648 = 2|E(G_4)| = \sum_{i=0}^{\infty} i f_i \geq 3f_3 + 4f_4 + 6(F_{\Pi} - (f_3 + f_4)).$$

and after manipulation we get

$$3f_3 + 2f_4 \geq 6\chi + 818. \quad (3)$$

Lemma 2.4 implies

$$f_4 \leq \sum_{i=0}^6 m_i a_i = a_0 + a_1 + \sum_{i=0}^6 i a_i. \quad (4)$$

Using Proposition 2.2 for $n = 4$ on (4) we obtain

$$f_4 \leq a_0 + a_1 + 3a. \quad (5)$$

Note that for any n we have $a = T(G_n) - f_3$, substituting this equality into (5) yields

$$3f_3 + f_4 \leq a_0 + a_1 + 3 \cdot 108. \quad (6)$$

Combining the last inequality with Proposition 2.3 for $n = 4$ we get

$$3f_3 + f_4 \leq 54 + 3 \cdot 108. \quad (7)$$

Two times (7) gives an upper bound on $6f_3 + 2f_4$, while adding three times (2) to (3) bounds $6f_3 + 2f_4$ from below. Combining these inequalities gives

$$\begin{aligned} 18\chi + 818 + 3 \cdot 324 &\leq 2 \cdot (54 + 3 \cdot 108) \\ 18\chi &\leq -1054 \\ \chi &< -58. \end{aligned}$$

Relating $\chi < -58$ with the genus of G_4 gives $\gamma(G_4) \geq 30$. □

Using the method from the proof of Theorem 2.7 for bounding the genus of G_5 gives $\gamma(G_5) \geq 133$, which is the same as the bound from Proposition 2.1.

3 Upper bounds

The determination of the genus of G_n in general seems to be a very difficult problem. On the contrary, when we concentrate only on symmetric embeddings of G_n , it is possible to determine not only the symmetric genus, but also the complete set of genera of surfaces upon which G_n admits a symmetric embedding. This fact is our motivation for discussing several natural variants of genus range for groups and Cayley graphs which were not investigated before.

3.1 Genus range and symmetric genus range of groups and Cayley graphs

We start by presenting definitions of symmetric and strongly symmetric embeddings of a Cayley graph. We follow [2, Chapter 11] to call an embedding of a Cayley graph G of a group Γ *symmetric* if the natural action of Γ by left-multiplication on the vertices of G can be extended to an action on the underlying surface. An embedding of a Cayley graph is called *strongly symmetric* if it is symmetric and the extended action preserves orientation of the surface. We introduce the *symmetric genus range of a Cayley graph G* as the set of genera of surfaces upon which G admits a symmetric embedding and *strong symmetric genus range of G* as the set of all genera of surfaces upon which G has a strong symmetric embedding. Note that the symmetric genus range and strong symmetric genus range parameters are analogous to the genus range parameter, thus extending the

correspondence between symmetric and all embeddings beyond the well-known (strong) symmetric genus of a Cayley graph.

Our main result in this direction is Theorem 3.2 which completely determines the strong symmetric genus range and symmetric genus range of G_n . Additionally, the theorem shows that, unlike the genus range, the (strong) symmetric genus range can contain arbitrarily large gaps.

For a set X of elements of a group, denote by \tilde{X} the union of elements of X and their inverses. Recall that a Cayley map of a Cayley graph G with a generating set X is an embedding of G in which every local rotation induces the same cyclic order of \tilde{X} . The proof of Theorem 3.2 is based on the following correspondence between Cayley maps and symmetric embeddings, see Chapter 10, Theorem 4.1, and Chapter 11 of [2]. An embedding of a Cayley graph G is strongly symmetric if and only if it is a Cayley map of G . An embedding of a Cayley graph of a group Γ with generating set X is symmetric, but not strongly symmetric, if and only if there is an index-two subgroup Γ' of Γ such that the local rotations of all vertices corresponding to Γ' induce the same cyclic order of \tilde{X} and the local rotations of all vertices corresponding to $\Gamma - \Gamma'$ induce the reverse cyclic order. Therefore, the problem of determining the strong symmetric genus range of G_n is equivalent with the problem of determining the genera of all Cayley maps of G_n . Moreover, if Γ does not have an index-two subgroup, then every symmetric embedding of G is strongly symmetric. Consequently, the fact that Z_3^n does not have an index-two subgroup for any nonnegative integer n implies that the symmetric genus range and the strong symmetric genus range of G_n coincide.

Recall that a generating set X of a group Γ is *irredundant* if no proper subset of X generates Γ . We call a Cayley graph *irredundant* if it is generated by an irredundant generating set of the group. Let B_n denote the bouquet of n circles; that is, a single vertex incident with n loops. It is well known that every Cayley graph of a group Γ with generating set not containing involutions is the derived graph of B_n for some integer n and a (regular) voltage assignment in Γ , see [9]. The derived embedding Π' is a Cayley map of G_n , whose genera was determined by [3]. In the case of irredundant Cayley graphs of Z_p^n it is given by formula

$$\gamma(\Pi') = 1 + \frac{|Z_p^n|}{2} \left(n - 1 - \sum_{i=1}^t \frac{1}{m_i} \right), \quad (8)$$

where t is the number of faces of Π and m_i , the *period* of the i -th face of Π , is the group order of the sum of group elements (voltages) assigned to the edges on the boundary of the face. All non-zero elements of the voltage group Z_p^n have order p . The following lemma characterises the possible periods of an embedding of B_n with voltages in Z_p^n such that the derived graph is an irredundant Cayley graph of Z_p^n .

Lemma 3.1. *Let Π be an embedding of B_n with a voltage assignment from Z_p^n for some odd prime p and positive integer n such that the derived graph is an irredundant Cayley graph of Z_p^n . If Π has one face, then the period of that face is 1. If Π has at least two faces, then the period of every face of Π is p .*

Proof. Since the generating set is irredundant, the order of every voltage is strictly greater than 1 and the voltages are pairwise independent. As the order of every element of Z_p^n is p , it follows that the order of every voltage is exactly p . Clearly, if Π has precisely one face, then every edge is traversed twice by the face, once in each direction, implying that the period is the group identity. Suppose that Π has at least two faces. If F is a face of Π , then the boundary of F contains an edge e that is traversed precisely once by F . The voltage assigned to e has order p and is independent from all other voltages assigned to the edges of F , thus the period of F is at least p . The fact that Z_p^n does not contain elements of order strictly greater than p implies that the period of F is exactly p . \square

Theorem 3.2. *Let G be an irredundant Cayley graph of Z_p^n for some prime p and a positive integer n . Then the symmetric genus range and the strong symmetric genus range of G coincide and are given by*

$$\{1 + p^{n-1}[(n-1)(p-1) - 2]/2 + gp^{n-1}; g \text{ is an integer such that } 0 \leq g < n/2\}.$$

Proof. It is not difficult to see that any Cayley map of G is the derived embedding of an embedding of B_n with voltages from Z_p^n . Clearly, B_n is a planar graph, the maximum genus of B_n is $\lfloor n/2 \rfloor$, and Interpolation theorem for orientable surfaces implies that B_n has a cellular embedding in the orientable surface of genus g if and only if $0 \leq g \leq \lfloor n/2 \rfloor$. Suppose that an embedding Π of B_n has at least two faces. By Lemma 3.1 the period of each face is p and therefore the genus of the derived embedding is determined by the number of faces of the base embedding alone. To calculate the genus of the derived embedding Π' , we can substitute f/p for the sum in (8), where f is the number of faces of Π . Additionally, expressing f from the Euler formula for Π we get $f = n + 1 - 2g$ and again substituting gives $\gamma(\Pi') = 1 + p^{n-1}[(n-1)(p-1) - 2 + 2g]/2$. A straightforward calculation shows that if the embedding of B_n has one face, then the genus of the derived embedding coincides with the largest value in the previous case. \square

As a consequence of Theorem 3.2, we get that the strong symmetric genus range of a Cayley graph can contain arbitrarily large gaps, as opposed to the genus range, which is always a contiguous interval by Interpolation theorem for orientable surfaces [8].

In particular, the lowest of genera of Cayley maps of G_n gives an upper bound on $\gamma(G_n)$.

Theorem 3.3. *Let n be a nonnegative integer. Then $\gamma(G_n) \leq 1 + 3^{n-1}(n-2)$.* \square

For $n = 3, 4$, and 5 , Theorem 3.3 gives $\gamma(G_3) \leq 10$, $\gamma(G_4) \leq 55$, and $\gamma(G_5) \leq 244$.

Concerning the genera of (strong) symmetric embeddings of Cayley graphs, a large part of the existing results deal with a determination of the genus – the minimum integer in the symmetric genus range. The focus is usually on a specific group or a family of groups, or a specific surface, see for example [7, 12]. Another important direction in the study of symmetric embeddings of Cayley graphs is aimed at regular maps arising from Cayley maps, see for example [20]. The following problem offers a slightly different perspective on symmetric embeddings of Cayley graphs. Moreover, Theorem 3.2 indicates that this problem may be more approachable in the case of irredundant Cayley graphs of groups with relatively simple structure such as Z_p^n .

Problem 4. *For a given Cayley graph G , determine the symmetric genus range and the strong symmetric genus range of G .*

The genus range of a graph G is the set of integers g such that G admits a cellular embedding in the orientable surface of genus g . By the Interpolation theorem for orientable surfaces ([8]), an integer g lies in the genus range of G if and only if $\gamma(G) \leq g \leq \gamma_M(G)$, where $\gamma_M(G)$ is the maximum genus of the graph G . However, for groups and their Cayley graphs, only the minimum genus parameter was introduced: the minimum genus of a group Γ is the minimum among genera of all Cayley graphs of Γ . Therefore, we formally propose the concept of genus range of a group. For a group Γ , the *genus range of Γ* is the set of all integers g such that there is a Cayley graph for Γ having a cellular embedding in the orientable surface of genus g . The maximum value in the genus range of a group Γ is called the *maximum genus of Γ* and is denoted by $\gamma_M(\Gamma)$. Theorem 3.2 suggests that the problem of determining the genus range of a group may have different characteristics when restricted to (strong) symmetric embeddings of irredundant Cayley graphs of the group. Therefore, we introduce also the symmetric and irredundant variants of the genus range of a group. For a group Γ , the *sharp genus range of Γ* is the set of all integers g such that there is an irredundant Cayley graph for Γ having a cellular embedding in the orientable surface of genus g . The maximum value in the sharp genus range of a group Γ is called the *sharp*

maximum genus of Γ and is denoted by $\gamma'_M(\Gamma)$. The (strong) symmetric genus range of Γ is the set of integers g such that there is a Cayley graph for Γ having a (strong) symmetric cellular embedding in the orientable surface of genus g . The sharp (strong) symmetric genus range of Γ is the set of integers g such that there is an irredundant Cayley graph for Γ having a (strong) symmetric cellular embedding in the orientable surface of genus g . Nonorientable analogues of the genus range parameters of a group may be introduced straightforwardly.

A graph G is called *upper-embeddable* if its maximum genus reaches the natural upper bound $\lfloor \beta(G)/2 \rfloor$, where $\beta(G)$ is the cycle rank of G . Equivalently, G is upper-embeddable if it has an embedding with one face (if its cycle rank is even) or with two faces (if its cycle rank is odd). While the calculation of the minimum genus of a group does not depend on the precise definition of the arising Cayley graphs, in the case of genus range and maximum genus the involutions (elements of order 2) in the generating sets need to be treated specifically. It is customary to define the Cayley graph to have cycles of length 2 corresponding to involutions and to define the *alternative, or reduced, Cayley graph* in which every cycle of length two is replaced by a single edge, see for instance [9]. Nedela and Škoviera [14] proved that every Cayley graph is upper-embeddable and that a reduced Cayley graph G is upper-embeddable unless the generating set consists from two elements r and s such that $r^2 = s^3 = 1$ and $|V(G)| \geq 18$, in which case an exact formula for the maximum genus is given.

The restriction to irredundant generating sets in the definition of genus of a group is justified by following observation: if X and X' are generating sets for a group Γ such that $X \subsetneq X'$, then the Cayley graph G of Γ generated by X is a subgraph of the Cayley graph G' generated by X' and consequently $\gamma(G) \leq \gamma(G')$. Without the restriction to irredundant generating sets also in the definition of the genus range, the whole group (without the identity element) may be chosen as the generating set, giving rise to a graph on $|\Gamma|$ vertices with $|\Gamma|(|\Gamma| - 1)/2$ edges. Thus the calculation of the maximum genus of Γ reduces to finding a generating set of Γ with least involutions in the case of reduced Cayley graphs and is trivial otherwise. The problem of determining the sharp maximum genus of a group reduces to finding an irredundant generating set X such that $|X| - i_X$ is maximized, where i_X is the number of involutions in X . In this context it would be interesting to know whether there is a group with non-upper-embeddable reduced Cayley graph for any irredundant generating set.

Since Z_p^n have essentially only one irredundant generating set, we get the next results.

Theorem 3.4. *For any odd prime p , the sharp maximum genus of Z_p^n is given by*

$$\gamma'_M(Z_p^n) = \lfloor ((n-1)p^n - 1)/2 \rfloor.$$

□

Theorem 3.5. *For any odd prime p , the sharp symmetric genus range and the sharp strong symmetric genus range of Z_p^n coincide and are given by*

$$\{1 + p^{n-1}[(n-1)(p-1) - 2]/2 + gp^{n-1}; g \text{ is an integer such that } 0 \leq g < n/2\}.$$

□

Theorem 3.6. *The sharp genus range of Z_3 , Z_3^2 , and Z_3^3 is equal to $\{0\}$, $\{1, 2, 3, 4\}$, and $\{g; 7 \leq g \leq 26\}$, respectively.*

□

The following problem suggests itself for both redundant and irredundant Cayley graphs of the group.

Problem 5. *For a given group Γ , determine the genus range, the (strong) symmetric genus range, sharp genus range, and sharp (strong) symmetric genus range of Γ .*

In spite of the Interpolation theorem for orientable surfaces and Theorem 3.2 it is natural to ask whether the genus range and the (strong) symmetric genus range of a group and their sharp analogues can contain gaps.

Finally, note that while the case of G_n is probably the most difficult in determining the (non-symmetric) genus among the Cayley graphs of abelian groups, most likely it is one of the easiest for the sharp (strong) symmetric genus.

3.2 Computer search

The first author wrote a series of computer programs for experimenting with the embeddings of G_n . The second author wrote an independent program for checking the validity of results. The data are available at www.dcs.fmph.uniba.sk/~kotrbcik/suppl/ and this section briefly summarizes the main results.

We start by introducing the invariant used to distinguish nonisomorphic embeddings of a graph. A *face distribution* of an embedding Π is the sequence $\{f_i\}$, where f_i is the number of faces of Π with length i . The concept of face distribution appears as 'region distribution' in [26], where all possible face distributions of K_5 and K_6 are presented, in the case of K_6 reporting an earlier work [11]. An *extended face distribution* of an embedding Π is the face distribution of Π together with the sequence $\{r_i\}$, where r_i is the number of faces of length i that contain some vertex more than once. Clearly, if two embeddings of the same graph have different extended face distribution, then they are nonisomorphic.

Theorem 3.7. *For the genus of G_4 we have $\gamma(G_4) \leq 37$. Moreover, there are more than 10.000 nonisomorphic embeddings of G_4 into the orientable surface of genus 37 with pairwise distinct extended face distributions.* \square

An embedding of G_4 in the orientable surface of genus 37 was obtained by computer search; the extended face distribution of the embedding is presented in Table 3, where a face is called repetitive if it contains some vertex more than once. The rotation schemes for more than 10.000 nonisomorphic embeddings of G_4 in the orientable surface of genus 37 and their extended face distributions can be found on the web pages containing the supplementary material. For the sake of completeness we present the corresponding rotations scheme of one such embedding in Table 4.

length of the face	3	4	5	6	7	8	9
number of faces	88	59	8	10	2	2	2
number of repetitive faces	0	0	0	0	2	0	0

Table 3: Extended face distribution of an embedding of G_4 with genus 37 presented in Table 4.

The problem of determining the complete genus distribution of a graph G asks for the number of embeddings of G in every surface, where two embeddings are considered to be different if their rotation schemes differ. Therefore, the following theorem does not take into account any symmetries of G_2 or the embedding.

Theorem 3.8. *The embedding range of G_2 is $[1, 5]$, that is, G_2 admits a cellular embedding into the surfaces of genus 1, 2, 3, 4, and 5, and the complete genus distribution is given in Table 5. In particular, there are 330 genus embeddings into torus with only 7 distinct extended face distributions, presented in Table 6, and 46.908 embeddings in the double torus with 146 distinct extended face distribution.* \square

On the web pages containing the supplementary material we list the rotations schemes and extended face distributions of all embeddings of G_2 with genus at most two. Furthermore, we

0000: 0200, 0100, 0001, 0002, 0020, 1000, 2000, 0010	2111: 2112, 2211, 2011, 2121, 2101, 0111, 1111, 2110
1000: 1010, 1020, 1200, 1100, 2000, 0000, 1002, 1001	0211: 1211, 0210, 0212, 0011, 0111, 0221, 0201, 2211
2000: 1000, 2100, 2200, 2002, 2001, 2020, 2010, 0000	1211: 2211, 1210, 1011, 1111, 1212, 1201, 1221, 0211
0100: 0110, 0120, 1100, 0102, 0101, 0000, 0200, 2100	2211: 2011, 2111, 2212, 2210, 1211, 0211, 2201, 2221
1100: 1102, 2100, 1000, 1200, 0100, 1120, 1110, 1101	0021: 0011, 0001, 0221, 0020, 0022, 2021, 1021, 0121
2100: 2120, 2110, 0100, 2200, 2000, 1100, 2102, 2101	1021: 0021, 2021, 1121, 1020, 1022, 1221, 1011, 1001
0200: 0000, 0210, 0220, 0202, 0201, 1200, 2200, 0100	2021: 2020, 2001, 2011, 2221, 2121, 1021, 0021, 2022
1200: 1201, 1202, 1100, 1000, 1220, 1210, 2200, 0200	0121: 0101, 0122, 0120, 0111, 0021, 1121, 2121, 0221
2200: 2000, 2100, 0200, 1200, 2210, 2201, 2202, 2220	1121: 1021, 2121, 0121, 1101, 1111, 1221, 1122, 1120
0010: 0011, 0110, 0210, 0000, 2010, 1010, 0020, 0012	2121: 2120, 2101, 0121, 1121, 2021, 2221, 2111, 2122
1010: 1000, 1012, 1020, 0010, 2010, 1011, 1210, 1110	0221: 0222, 0220, 0021, 0201, 0211, 0121, 2221, 1221
2010: 2012, 2011, 1010, 0010, 2000, 2020, 2110, 2210	1221: 2221, 1222, 1121, 1021, 1220, 1211, 1201, 0221
0110: 0010, 0112, 0111, 0120, 0100, 2110, 1110, 0210	2221: 2220, 2222, 1221, 0221, 2121, 2021, 2211, 2201
1110: 2110, 1111, 1100, 1120, 1112, 1010, 1210, 0110	0002: 0000, 0001, 1002, 2002, 0012, 0102, 0202, 0022
2110: 2112, 2111, 1110, 0110, 2100, 2120, 2210, 2010	1002: 0002, 1001, 1000, 1022, 1012, 1102, 1202, 2002
0210: 0200, 0010, 0110, 1210, 0212, 0211, 2210, 0220	2002: 2022, 2012, 0002, 1002, 2202, 2102, 2001, 2000
1210: 1211, 2210, 1200, 1220, 1212, 0210, 1110, 1010	0102: 2102, 0202, 0002, 0112, 0122, 0101, 0100, 1102
2210: 2211, 2212, 2010, 2110, 0210, 2220, 2200, 1210	1102: 1202, 1002, 1112, 1122, 1100, 1101, 2102, 0102
0020: 0220, 0120, 0010, 1020, 2020, 0000, 0022, 0021	2102: 2202, 2101, 2100, 2122, 2112, 0102, 1102, 2002
1020: 1022, 1021, 1120, 1220, 1000, 2020, 0020, 1010	0202: 1202, 0201, 0200, 0222, 0002, 0102, 0212, 2202
2020: 2220, 2120, 0020, 1020, 2010, 2000, 2021, 2022	1202: 1002, 1102, 1200, 1212, 1222, 1201, 0202, 2202
0120: 0110, 0121, 0122, 0020, 0220, 2120, 1120, 0100	2202: 2200, 2201, 2102, 2002, 1202, 0202, 2212, 2222
1120: 0120, 2120, 1220, 1020, 1121, 1122, 1110, 1100	0012: 1012, 0011, 0010, 0022, 0212, 0112, 0002, 2012
2120: 2121, 2122, 2020, 2220, 1120, 0120, 2110, 2100	1012: 0012, 2012, 1212, 1112, 1002, 1022, 1010, 1011
0220: 0020, 2220, 1220, 0221, 0222, 0200, 0210, 0120	2012: 0012, 2002, 2022, 2112, 2011, 2010, 2212, 1012
1220: 1222, 1210, 1200, 1020, 1120, 2220, 0220, 1221	0112: 0012, 0212, 2112, 1112, 0111, 0110, 0122, 0102
2220: 0220, 1220, 2120, 2020, 2200, 2222, 2221, 2210	1112: 1102, 1012, 1212, 1111, 0112, 2112, 1110, 1122
0001: 0201, 0021, 0011, 2001, 1001, 0002, 0000, 0101	2112: 2012, 2212, 2111, 2110, 1112, 0112, 2102, 2122
1001: 1000, 1002, 0001, 2001, 1201, 1101, 1021, 1011	0212: 0012, 0222, 0211, 0210, 1212, 2212, 0202, 0112
2001: 2201, 1001, 0001, 2011, 2021, 2000, 2002, 2101	1212: 1211, 1112, 1012, 2212, 0212, 1210, 1222, 1202
0101: 1101, 0201, 0001, 0100, 0102, 0121, 0111, 2101	2212: 1212, 2012, 2210, 2211, 2112, 2222, 2202, 0212
1101: 2101, 1102, 1100, 1111, 1121, 1001, 1201, 0101	0022: 1022, 2022, 0021, 0020, 0002, 0222, 0012, 0122
2101: 2100, 2102, 2201, 2001, 1101, 0101, 2111, 2121	1022: 1012, 1002, 2022, 0022, 1122, 1222, 1021, 1020
0201: 2201, 0211, 0221, 0001, 0101, 1201, 0200, 0202	2022: 0022, 1022, 2122, 2222, 2012, 2002, 2020, 2021
1201: 1101, 1001, 2201, 1202, 1221, 1211, 1200, 0201	0122: 0222, 2122, 1122, 0022, 0120, 0121, 0102, 0112
2201: 0201, 1201, 2001, 2101, 2202, 2200, 2221, 2211	1122: 1112, 1120, 1121, 1222, 1022, 0122, 2122, 1102
0011: 2011, 0001, 0021, 0111, 0211, 0010, 0012, 1011	2122: 2120, 2121, 2112, 2102, 1122, 0122, 2222, 2022
1011: 0011, 1012, 1001, 1021, 1111, 1211, 1010, 2011	0222: 0212, 0022, 0202, 0220, 0221, 1222, 2222, 0122
2011: 2021, 2001, 0011, 1011, 2010, 2012, 2111, 2211	1222: 2222, 0222, 1202, 1212, 1220, 1022, 1122, 1221
0111: 0121, 0110, 0112, 1111, 2111, 0101, 0211, 0011	2222: 0222, 1222, 2221, 2220, 2202, 2212, 2022, 2122
1111: 1112, 1211, 1011, 1121, 1101, 1110, 2111, 0111	

Table 4: Rotation scheme for an embedding of G_4 with genus 37.

genus	0	1	2	3	4	5	6
# embeddings	0	330	46.908	1.385.214	6.516.564	2.128.680	0

Table 5: Complete genus distribution of G_2 .

provide all distinct face distributions and the corresponding rotation schemes for embeddings of genus at most two. Perhaps surprisingly, G_2 embedded in the torus admits only 7 distinct extended face distributions, they are listed in Table 6 along with the number of such embeddings. Three of these distributions contain exactly one repetitive face and four of them do not contain a repetitive face. Finally, note that the two embeddings with 9 quadrangles are mirror images of each other.

length of the face	3	4	5	6	length of the repetitive face	frequency
number of faces	6	2	0	0	10	36
number of faces	5	2	1	0	8	144
number of faces	4	2	2	1	6	72
number of faces	4	3	0	2	–	36
number of faces	4	1	4	0	–	36
number of faces	6	0	0	3	–	4
number of faces	0	9	0	0	–	2

Table 6: All distinct extended face distributions of G_2 embedded in the torus.

Theorem 3.9. *There are at least 1319 genus embeddings of G_3 with pairwise distinct extended face distributions.* \square

The rotation schemes for the nonisomorphic genus embeddings of G_3 can be found on the web pages containing the supplementary material. Table 7 contains several particularly interesting

face distributions of genus embeddings of G_3 ; the corresponding embeddings can be also found as a separate part of the supplementary material. Although all these embeddings have all faces nonrepetitive, it is interesting that for all of them except the first two, there is an embedding with the same face distribution and one of the longest faces repetitive. Note also that the last embedding in Table 7 has the same face distribution as the embedding constructed in [13] to show that $\gamma(G_3) \leq 7$.

length of the face	3	4	5	6	length of the remaining face
number of faces	22	12	0	8	–
number of faces	24	7	4	7	–
number of faces	24	9	0	9	–
number of faces	26	9	0	6	12
number of faces	27	6	0	8	9
number of faces	27	8	0	6	13
number of faces	27	9	0	5	15

Table 7: Face distributions of some of the 1319 genus embeddings of G_3 from Theorem 3.9.

Note that G_n has in general $\prod_{v \in V(G_n)} (\deg(v) - 1)! = [(2n - 1)!]^{3^n}$ rotation schemes. In particular, G_3 and G_4 have $120^{27} \approx 10^{56}$, respectively $5040^{81} \approx 10^{299}$, rotation schemes, which makes exhaustive search infeasible even for G_3 .

3.3 Recursive construction

Let G'_n denote G_n with a loop attached to every vertex. Clearly, G_{n+1} is the derived graph of G'_n with respect to Z_3 with a non-zero element of Z_3 assigned to an edge e if and only if e is a loop. It is easily seen that if a loop bounds a face (of length 1) and the voltage assigned to the loop has order 3, then the face lifts to a triangle. Therefore, if every loop bounds a face and a non-zero element of Z_3 is assigned to every loop, then the derived embedding has at least $|V(G'_n)|$ triangles. The main idea of the proof of Theorem 3.11 is that if we can embed loops inside faces of an embedding Π in such a way that every face contains either zero or at least two loops, then there is a voltage assignment to loops such that every face of Π with length at least 2 lifts to three faces. The fact that the loops can be distributed appropriately is captured by the following definition.

Definition 3.10. *Let Π be an embedding of a graph G . A face-covered partition of Π is a partition of the vertex set $V(G)$ into sets P_i , $i = 1, \dots, k$ such that each P_i contains at least two vertices and for any i there is a face of Π whose boundary contains all vertices from P_i .*

Our method does not rely on the fact that the base graph is G_n and we state the result in a more general form.

Theorem 3.11. *Let Π be an embedding of a graph G in an orientable surface S . Let G' denote G with a loop attached to every vertex. If Π admits a face-covered partition of G , then there is an embedding of G' in S and a voltage assignment from Z_3 to G' such that the derived graph is $G \square K_3$ embedded with $3F_\Pi + |V(G)|$ faces.*

Proof. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a face-covered partition of Π for G and let F_i be a face of Π that covers P_i for $i \in \{1, \dots, k\}$. We arbitrarily choose a preferred direction for each loop e and denote it by e ; the opposite direction is denoted by e^{-1} . For any vertex $v \in \mathcal{P}$, embed the loop e based at v into F_i in such a way that the rotation at v is $(ee^{-1}a\dots)$, where a is an arc traversed by F_i . Denote the resulting embedding by Π' . Denote by L the set of loops added to G and by F_L the

set of faces of length 1 bounded by a loop from L . We say that a face F of Π' *contains a loop* l if the boundary of F traverses l . Our goal is to prescribe a voltage assignment ζ from the arcs of G' to the elements of Z_3 such that the derived embedding has $3F_\Pi + |V(G)|$ faces. To achieve this property, we choose ζ such that the period of each face in F_L is 3 and the period of each face of Π' not in F_L is 1.

First note that any loop e from L lies on the boundary of two distinct faces in Π' , one of them is a face of length 1 bounded by e , and the second is a face of length at least 2. For any arc a of G not contained in G' let $\zeta(a) = 0$. Let F be a face of Π' not in F_L and containing a loop. The choice of \mathcal{P} and Π' implies that F contains at least two loops. If F contains two loops e and f , let $\zeta(e) = 1$ and $\zeta(f) = 2$. If F contains three loops e , f , and g , let $\zeta(e) = \zeta(f) = \zeta(g) = 1$. Finally, if F contains at least four loops, choose any two of them, say e and f , and let $\zeta(e) = 1$ and $\zeta(f) = 2$. By repeating the above process the value of ζ can be defined for all loops contained in F and consequently for all arcs of G' .

Now we show that the periods of all faces of Π' under ζ have the required values. The boundary of any face of F_L contains precisely one arc a corresponding to a loop in L and $\zeta(a) = 1$ or $\zeta(a) = 2$. In both cases the period of the face in Z_3 is 3. If F is a face of Π' not in F_L and not containing a loop, then all arcs on the boundary of F are assigned 0 by ζ and the period of F is 1. If F is a face of Π' not in F_L containing two or three loops, then clearly the period of the face is 1. The fact that the period of F is 0 if F contains at least four loops can be established by an easy inductive argument which is omitted.

Clearly, the derived graph of G' under ζ in Z_3 is $G \square K_3$. The embedding Π' has $F_\Pi + |V(G)|$ faces, F_Π of them with period 3 and $|V(G)|$ of them with period 1. Therefore, the derived embedding of Π' under ζ in Z_3 has $F_\Pi + |V(G)|$ faces, which completes the proof. \square

Our next aim is to show that every minimum-genus embedding of G_n admits a face-covered partition. The main idea of our method of constructing face-covered partitions is that if we take a matching in G_n , then every pair of matched vertices is covered by some face. As $G_n - v$ contains a perfect matching for any vertex v , the problem reduces to covering the exceptional vertex v . To cover this vertex we then use the fact that the embedding has a face of length at most 5. In the proofs of the following two auxiliary results we use the fact that the vertices of G_n can be bijectively identified with words of length n over $\{0, 1, 2\}$ with two vertices being adjacent if and only if their representations differ at exactly one position.

Lemma 3.12. *Let S be a set of three pairwise adjacent vertices of G_n for some $n \geq 1$. Then $G_n - S$ has a perfect matching. Consequently, $G_n - v$ has a perfect matching for any vertex v .*

Proof. First observe that if $G_n - S$ has a perfect matching for any triangle S , then $G_n - v$ has a perfect matching for any vertex v , as every vertex lies in some triangle. We proceed by induction on n . For $n = 1$ the claim is obvious. For $n \geq 2$ we show how to construct the desired matching. From the fact that S forms a triangle it follows that there is a unique position such that the representations of the vertices of S pairwise differ only in this position. Restricting $G_n - S$ to all but this one position yields three disjoint copies of G_{n-1} , each of them with one vertex removed. By the induction hypothesis the copies of G_{n-1} have a perfect matching; union of these perfect matchings is a perfect matching of $G_n - S$. \square

Proposition 3.13. *Let Π be an embedding of G_n that contains a face of length at most 5. Then Π admits a face-covered partition of G .*

Proof. We distinguish three cases according to the length of the shortest face of Π .

i) *The length of a shortest face of Π is 3.* Let F be a face of length 3 and let S be the set of vertices incident with S . By Lemma 3.12, there is a perfect matching M of $G_n - S$. Denote the edges of M by m_1, \dots, m_k and let $P_i = \{u_i, v_i\}$, where u_i and v_i are endpoints of m_i for $i \in \{1, \dots, k\}$. Finally, let $P_{k+1} = S$. Since every edge m_i is traversed by some face of Π , for any i , $1 \leq i \leq k$,

there is a face of Π that covers both vertices of P_i . Moreover, the vertices of P_{k+1} are covered by F . It follows that the system of sets P_i for $i \in \{1, \dots, k+1\}$ is a face-covered partition of Π .

ii) *The length of a shortest face of Π is 4.* Let F be a face of length 4 in Π . Without loss of generality suppose that the vertices of F are represented by $00x, 10x, 11x$, and $01x$, where x is arbitrary, but fixed word of length $n-2$ over $\{0, 1, 2\}$. Consider the graph $G' = G_n - \{abx; a, b \in \{0, 1, 2\}\}$. Restricting G' to positions $2, \dots, n$ yields three disjoint copies of G_{n-1} , each of them with the triangle at position 2 removed. By Lemma 3.12, each copy of G_{n-1} with a triangle removed admits a perfect matching; denote by M the union of these perfect matchings. We construct a face-covered partition of G_n using M and a partition covering the three removed triangles. Suppose that the edges of M are $u_1v_1, u_2v_2, \dots, u_kv_k$, where $k = 3^{n-1} - 3$. Let $P_i = \{u_i, v_i\}$ for $i \in \{1, \dots, k\}$. Clearly, these sets cover all vertices of M . To cover the vertices of the removed triangles, let $P_{k+1} = \{00x, 10x, 11x\}$, $P_{k+2} = \{20x, 21x\}$, $P_{k+3} = \{01x, 02x\}$, and $P_{k+4} = \{12x, 22x\}$. Since the set P_{k+1} is covered by the face F and any set P_i for $i \neq k+1$ contains exactly two vertices joined by an edge, the system of sets P_i , $i \in \{1, \dots, k+4\}$ forms the desired face-covered partition.

iii) *The length of a shortest face of Π is 5.* Every pentagon in G_n has form $aabab^{-1}$ for some generators of G_n a and b . In particular, every pentagon contains vertices of some triangle S . To get the desired face-covered partition it suffices to take the face-covered partition for Π covering the vertices of S which is constructed in the proof of case i). \square

Proposition 3.14. *Every embedding of G_n with genus at most $1 + \lceil 3^{n-1}(2n-3)/2 \rceil$ contains a face of length at most 5. In particular, every minimum-genus embedding of G_n contains a face of length at most 5.*

Proof. If an embedding Π of G_n has the length of a shortest face at least 6, then Π contains at most $n3^{n-1}$ faces. Using Euler formula we get that the genus of Π is at least $1 + \lceil 3^{n-1}(2n-3)/2 \rceil$, which justifies the first claim. By Theorem 3.3 $\gamma(G) \leq 1 + n3^{n-1} - 2 \cdot 3^{n-1}$. Therefore, the inequality $1 + n3^{n-1} - 2 \cdot 3^{n-1} < 1 + \lceil 3^{n-1}(2n-3)/2 \rceil$ proves the the second claim. \square

Theorem 3.15. *For any $n \geq 1$ we have $\gamma(G_{n+1}) \leq 3\gamma(G_n) + 3^n - 2$. Consequently, for $n \geq 5$ we have*

$$\gamma(G_n) \leq 3^{n-4}[\gamma(G_4) + 27n - 109] + 1 \leq 3^{n-2}[3n - 8] + 1.$$

Proof. Let Π be a genus embedding of G_n for some $n \geq 2$. By Proposition 3.13 and Proposition 3.14, the embedding Π has a face-covered partition \mathcal{P} . By Theorem 3.11 applied to Π and \mathcal{P} , there is an embedding Π' of G_{n+1} with $3F_\Pi + 3^n$ faces. Using Euler formula on the number of faces of Π and Π' and the number of vertices and edges of the corresponding graphs gives the first claim. The first closed form from the second claim can be obtained by solving the recurrence relation $g_4 = \gamma(G_4)$ and $g_{n+1} = 3g_n + 3^n - 2$ for $n \geq 4$. The second closed form follows from Theorem 3.7 as $\gamma(G_4) \leq 37$. \square

Corollary 3.16. *For the genus of G_5 we have $\gamma(G_5) \leq 190$.*

Proof. By Theorem 3.7, there is an embedding Π of G_4 with genus 37. The embedding Π contains a triangular face and therefore, by Proposition 3.13, it admits a face-covered partition \mathcal{P} . The result follows by applying Theorem 3.11 to Π and \mathcal{P} . \square

Proof of Theorem 1.1. The lower bounds follow from Theorem 2.7 and Proposition 2.1. The upper bounds are proved in Theorem 3.7 and Theorem 3.15. \square

Theorem 3.11 and Lemma 3.13 can be applied also to genus embeddings of G_2 and G_3 to construct low-genus embeddings of G_3 and G_4 , respectively. In these cases we get that the derived embeddings have genera 10 and 46, respectively, yielding $\gamma(G_3) \leq 10$ and $\gamma(G_4) \leq 46$. Note that both these bounds have been superseded by ad-hoc and computer-search methods of this paper.

Discussion

The paper improves the bounds on the genus of G_n by combining computational, recursive, combinatorial, and to a very limited extent also group-theoretic methods. Although at present the search for the genus of G_n seems to be intractable without new techniques, we hope that the gap $30 \leq \gamma(G_4) \leq 37$ is a challenge that will attract mathematicians and computer programmers alike. Any improvement on upper bounds for any G_n with $n \geq 4$ immediately yields better upper bounds for all G_m with $m > n$ by Theorem 3.15. On the other hand, it would be very desirable to have lower bounds on the genus of G_n that improve on the, in a sense trivial, bounds of Proposition 2.1.

Note that there are several related problems worth attacking that were not investigated in this paper, one of them being the determination of the non-orientable genus of G_n .

In our computations we have not considered the fact that G_n is highly symmetric. Therefore, some of the embeddings we have constructed in our search for low-genus embeddings of G_4 are pairwise isomorphic. It is possible that a certain speed-up may be achieved by considering only representatives of equivalence classes as is the case in other applications, such as [6].

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