RSA

Martin Stanek

Department of Computer Science
Comenius University
stanek@dcs.fmph.uniba.sk

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paddings

Introduction

- RSA – Ron Rivest, Adi Shamir, Leonard Adleman (1977)
- Clifford Cocks (1973), declassified in 1997
- one of the most frequently used public-key schemes
- encryption scheme / digital signature scheme

Initialization (key generation)

1. choose large, distinct primes p, q (e.g. 1024 bits long)
2. let $n = p \cdot q$ (public modulus)
3. choose e such that $\gcd(e, \varphi(n)) = 1$
   - φ is Euler's totient function
   - $\varphi(n) = (p-1)(q-1)$
4. compute d such that $e \cdot d \equiv 1 \pmod{\varphi(n)}$

- public key: (e, n); e public exponent
- private key: (d, n); d private exponent
- additional values are often stored as a part of the private key to speed up the private transformation

Encryption and decryption

- textbook/plain RSA
- encryption, decryption: $E, D : \mathbb{Z}_n \to \mathbb{Z}_n$:

$$E(m) = m^e \bmod n$$
$$D(c) = c^d \bmod n$$

- small example:
  - $p = 11$, $q = 19$, $n = 11 \cdot 19 = 209$, $\varphi(209) = 10 \cdot 18 = 180$
  - $e = 7$, $d = 7^{-1} \bmod 180 = 103$
  - public key: $(7, 209)$; private key: $(103, 209)$
  - let $m = 100$: encryption $E(100) = 100^7 \bmod 209 = 111$
  - decryption $D(111) = 111^{103} \bmod 209 = 100$

Basic facts from number theory 1

- Notation (for positive integer n):
  - $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$
  - $\mathbb{Z}_n^* = \{a \,;\; a \in \mathbb{Z}_n \;\&\; \gcd(a, n) = 1\}$
- Euler's totient function: $\varphi(n) = |\mathbb{Z}_n^*|$
  - $\varphi(8) = |\{1, 3, 5, 7\}| = 4$
  - $\varphi(p) = |\{1, 2, \ldots, p-1\}| = p - 1$ for prime p
  - $\varphi(p \cdot q) = (p-1)(q-1)$ for product of two distinct primes
- $a \equiv b \pmod{n} \;\Leftrightarrow\; n \mid (a - b)$

Basic facts from number theory 2

Lemma 1

Let $ka \equiv kb \pmod{n}$ *for positive integer* n *and integers* a, b, k. *Let* $\gcd(k, n) = 1$. *Then* $a \equiv b \pmod{n}$.

Proof.

$$ka \equiv kb \pmod{n} \implies n \mid k(a-b)$$

Since n and k are coprime, we have $n \mid (a-b)$ □

Basic facts from number theory 3

Lemma 2

Let $\mathbb{Z}_n^* = \{a_1, \ldots, a_{\varphi(n)}\}$. *Let* k *be an integer such that* $\gcd(k, n) = 1$.
Then $\{ka_1 \bmod n, \ldots, ka_{\varphi(n)} \bmod n\} = \mathbb{Z}_n^*$

Proof.

1. $\gcd(a_i, n) = 1,\ \gcd(k, n) = 1 \implies \gcd(ka_i, n) = 1$
   Hence $\{ka_1 \bmod n, \ldots, ka_{\varphi(n)} \bmod n\} \subseteq \mathbb{Z}_n^*$
2. $\gcd(k, n) = 1,\ ka_i \equiv ka_j \pmod{n} \implies a_i \equiv a_j \pmod{n}$ (Lemma 1)
   $\implies i = j$, and therefore all elements in the set are distinct.

□

Euler's theorem

Theorem 3 (Euler)

Let n be a positive integer. Then for an arbitrary integer a coprime to n:

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Proof.

Let $\mathbb{Z}_n^* = \{a_1, \ldots, a_{\varphi(n)}\}$, so $\mathbb{Z}_n^* = \{aa_1 \bmod n, \ldots, aa_{\varphi(n)} \bmod n\}$ (Lemma 2).
We compute the product of all elements:

$$\prod_{i=1}^{\varphi(n)} a_i \equiv \prod_{i=1}^{\varphi(n)} a \cdot a_i \equiv a^{\varphi(n)} \prod_{i=1}^{\varphi(n)} a_i \pmod{n}$$

Since $\gcd(a_i, n) = 1$, the product is coprime to n as well. Applying Lemma 1 we get $a^{\varphi(n)} \equiv 1 \pmod{n}$. □

Euler's theorem – remarks

- Fermat's little theorem:
  Let p be a prime, and a be an integer. If $p \nmid a$ then $a^{p-1} \equiv 1 \pmod{p}$.
- FLT is direct corollary of Euler's theorem:
  - $p \nmid a \Leftrightarrow \gcd(a, p) = 1$
  - $\varphi(p) = p - 1$
- Carmichael's function $\lambda(n)$: smallest positive integer such that $a^{\lambda(n)} \equiv 1 \pmod{n}$ for every integer a coprime to n
  - $\lambda(p) = p - 1$, $\lambda(p^l) = p^{l-1}(p-1)$ for prime p
  - $\lambda(n) = \operatorname{lcm}(\lambda(p_1^{l_1}), \lambda(p_2^{l_2}), \ldots, \lambda(p_k^{l_k}))$, where $n = p_1^{l_1} p_2^{l_2} \ldots p_k^{l_k}$ is a prime decomposition
  - generalization of Euler's theorem ($a^{\lambda(n)} \equiv 1 \pmod{n}$ for a coprime to n)
  - sometimes the RSA is specified in terms of $\lambda(n)$
  - $\lambda(p \cdot q) = \operatorname{lcm}(p-1, q-1)$

Correctness of RSA

Theorem 4 (Correctness of RSA)

Let E and D be the encryption and decryption functions in RSA scheme. Then

$$\forall m \in \mathbb{Z}_n : \ D(E(m)) = m.$$

Proof.

Case 1: $\gcd(m, n) = 1$; the most frequent case

$$\begin{aligned} D(E(m)) &= (m^e)^d \bmod n = m^{1+k\varphi(n)} \bmod n \\ &= m \cdot (\underbrace{m^{\varphi(n)}}_{1})^k \bmod n \qquad \text{(Euler's theorem)} \\ &= m \bmod n = m \end{aligned}$$

Correctness of RSA 2 (proof cont.)

Case 2: $\gcd(m, n) > 1$, rare event
(you can factorize n if this happens for $m \neq 0$)

- trivially valid if $m = 0$
- wlog we assume $m = m' \cdot p^l$ for $l \geq 1$ and $\gcd(m', n) = 1$
- $D(E(m)) = (m'p^l)^{ed} \bmod n = m' \cdot (p^{1+k\varphi(n)})^l \bmod n$ using case 1
- evaluating expression $p^{1+k\varphi(n)} \bmod n$:

$$
\begin{aligned}
p^{(q-1)} &\equiv 1 \pmod{q} \qquad \text{(FLT)} \\
p^{k(q-1)(p-1)} \equiv p^{k\varphi(n)} &\equiv 1 \pmod{q} \\
p^{k\varphi(n)} &= 1 + t \cdot q \quad \Rightarrow \quad p^{1+k\varphi(n)} = p + t \cdot n
\end{aligned}
$$

- therefore $p^{1+k\varphi(n)} \equiv p \pmod{n}$ and $D(E(m)) = m' \cdot p^l \bmod n = m$

□

Correctness – remarks

- E, D are two mutually inverse bijections
- some fixed points: $E(0) = 0$, $E(1) = 1$, $E(n-1) = n-1$

Implementation – “how” & efficiency issues

- modular exponentiation
- primality testing
- computing private exponent (modular inverse)
- choosing public exponent for efficiency
- improving performance of private transformation by Chinese remainder theorem

Modular exponentiation

- compute $a^k \bmod n$ for (positive) integers a, k, n
- note that $\underbrace{a \cdot a \cdot \ldots \cdot a}_{k} \bmod n$ is an exponential algorithm w.r.t. $|k|$

polynomial time algorithm:

$v = 1$
while $(k > 0)$
    if k is odd: $v = v \cdot a \bmod n$
    $a = a^2 \bmod n$
    $k = k/2$ // *integer division*
return v

example: $a^{21} \bmod n$

$(k, v, a)_{\text{before}}$	$(k, v, a)_{\text{after}}$
$(21, 1, a)$	$(10, a, a^2)$
$(10, a, a^2)$	$(5, a, a^4)$
$(5, a, a^4)$	$(2, a^5, a^8)$
$(2, a^5, a^8)$	$(1, a^5, a^{16})$
$(1, a^5, a^{16})$	$(0, a^{21}, a^{32})$

- other improvements: sliding window, Montgomery reduction

Choosing primes

- primes should be secret (otherwise an attacker can easily compute d)
- procedure: random choice of odd integer + primality testing
- density of primes:
  - $\pi(n)$ – number of primes less than or equal to n
  - Prime number theorem: $\pi(n) \sim n/\ln(n)$
  - bounds for $\pi(n)$, e.g. for $n \geq 55$: $n/(\ln n + 2) < \pi(n) < n/(\ln n - 4)$
  - these bounds can be used to estimate the probability that random odd number of certain length is a prime
- real experiment (average from 300 samples):

bit length	count
128	44
512	180
786	271
1024	353

Primality testing

- deciding primality is in **P**
  - AKS primality test (2002); slow, not used in practice
- probabilistic tests offer better performance: Miller-Rabin, Lucas, etc.
- Miller-Rabin test (and its variants or combination with other tests) is the most common choice
  - FIPS 186-4 Digital Signature Standard
  - openssl implementation, ...

Miller-Rabin test

- input: odd n; let $n - 1 = t \cdot 2^s$ for odd integer t
- n is *strong pseudoprime* to a base a (where $1 \leq a < n$) if:

$$a^t \equiv 1 \pmod{n} \quad \vee \quad \exists r \in \mathbb{Z}_s : a^{t \cdot 2^r} \equiv -1 \pmod{n}$$

- if n is prime, then n is strong pseudoprime to every base
- if n is composite, then the probability that n is strong pseudoprime to a random base is at most $1/4$
  - probability of error after k independent choices of the base is 4^{-k}
  - much smaller for most n
- repeated squaring for the second part of strong pseudoprime property
- some insight into the property:
  - if n is prime: $a^{n-1} \equiv a^{t \cdot 2^s} \equiv 1 \pmod{n}$ for all a not divisible by n (FLT)
  - if n is prime: 1 has exactly two square roots modulo n;
    $x^2 \equiv 1 \pmod{n} \implies n \mid (x+1)(x-1) \implies x \equiv \pm 1 \pmod{n}$

Computing private exponent

- $d = e^{-1} \bmod \varphi(n)$
- Extended Euclidean algorithm
  - input: integers a, b
  - output: $\gcd(a, b)$, integers x, y such that $xa + yb = \gcd(a, b)$
  - remark: returning gcd is redundant if x, y are known
  - simple recursive version (for integers $a, b \geq 0$):

    EEA(a,b)
    if $b = 0$: return$(a, 1, 0)$
    $(d, x, y) \leftarrow$ EEA$(b, a \bmod b)$
    return $(d, y, x - y \cdot \lfloor a/b \rfloor)$
- EEA$(e, \varphi(n)) \mapsto (1, x, y)$: $xe + y\varphi(n) = 1 \implies d = x \bmod \varphi(n)$

Choosing public exponent

- improving performance: small public exponent
- common choice $e = 65537 = (1000 \ldots 0001)_2$
  - it is a prime and with high probability coprime to $\varphi(n)$
  - if $e = 65537$ is desired, we can test $\gcd(e, p-1) = 1$ (and for q as well) while generating the primes
  - nice binary representation (short and small number of ones)

Chinese remainder theorem

- used in various constructions and implementations with modular arithmetic

Theorem 5 (CRT)

Let $n_1, \ldots, n_k$ are pairwise coprime positive integers. Then the following system of congruences (where $a_1, \ldots, a_k$ are arbitrary integers):

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ &\ldots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

has a solution. Additionally, all solutions of the system are mutualy congruent modulo $N = n_1 \cdot \ldots \cdot n_k$.

Chinese remainder theorem – proof

Proof.

1. Let $N_i = N/n_i$ for $i = 1, \ldots, k$, and $M_i = N_i^{-1} \bmod n_i$.
   Solution $x = \sum_{i=1}^{k} a_i N_i M_i$ can be easily verified (for $j \in \{1, \ldots, k\}$):

$$x \equiv a_j \underbrace{N_j M_j}_{1 \bmod n_j} + \sum_{\substack{i=1 \\ i \neq j}}^{k} a_i M_i \underbrace{N_i}_{0 \bmod n_j} \equiv a_j \pmod{n_j}.$$

2. Let x and x' are two solutions. Therefore

$$\begin{aligned} x &\equiv x' \pmod{n_1} \\ &\ldots \\ x &\equiv x' \pmod{n_k} \end{aligned} \qquad \Rightarrow \quad \forall i : \; n_i \mid (x - x')$$

   Since n_i are pairwise coprime we have $N \mid (x - x')$.

□

Corollary of the CRT

- $n = p \cdot q$, $\gcd(p, q) = 1$

$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv a \pmod{q} \end{aligned} \qquad \Longleftrightarrow \qquad x \equiv a \pmod{n}$$

- ($\Rightarrow$) value a is a solution;
  according to the CRT, if x is also a solution, then $x \equiv a \pmod{n}$
- ($\Leftarrow$) trivial: $x = a + t \cdot pq \;\Rightarrow\; x = a + (tq) \cdot p \;\Rightarrow\; x \equiv a \pmod{p}$
  (similarly for q)

Optimization of private transformation $D(c)$

- idea: instead of $c^d \bmod n$ compute $c^d \pmod p$, $c^d \pmod q$ and combine results using the CRT
- for unknown m:

$$\begin{aligned} m &\equiv c^d \pmod p \\ m &\equiv c^d \pmod q \end{aligned} \iff m \equiv c^d \pmod n$$

we can obtain m as follows:

$$m = m_p \cdot q(q^{-1} \bmod p) + m_q \cdot p(p^{-1} \bmod q) \bmod n$$

where $m_p = c^d \bmod p = c^{d \bmod (p-1)} \bmod p$ and similarly $m_q = c^d \bmod q = c^{d \bmod (q-1)} \bmod q$

- two "half-size" modular exponentiations are faster than one "full-size"
- p, q – part of private key; pre-computed inverses

Optimization in real world

- private key includes: $p, q, d_p = d \bmod (p-1), d_q = d \bmod (q-1), q_{\text{inv}} = q^{-1} \bmod p$
- computation of $D(c)$:
  1. $m_p = c^{d_p} \bmod p$, $m_q = c^{d_q} \bmod q$
  2. $m = m_q + q(q_{\text{inv}}(m_p - m_q) \bmod p)$

- the correctness of the result can be easily verified by checking:

$$m \bmod p = m_p$$
$$m \bmod q = m_q$$

and noticing that $0 \leq m < pq$

Real-world implementation (playing with openssl)

- creating 2048-bit RSA instance:
  `openssl genrsa -out myrsa.pem 2048`
- What is inside?
  `openssl rsa -in myrsa.pem -text`
  - modulus: n
  - publicExponent: e
  - privateExponent: d
  - prime1, prime2: p, q
  - exponent1, exponent2: $d \bmod (p-1)$, $d \bmod (q-1)$
  - coefficient: $q_{\text{inv}} = q^{-1} \bmod p$
- extracting public key:
  `openssl rsa -in myrsa.pem -pubout -out myrsa-pub.pem`

PKCS#1 v1.5 padding

- why padding at all?
  - to randomize encryption (plain RSA is deterministic)
  - prove the security of the scheme (OAEP)
- PKCS#1 v1.5 (still used, potential implementation problems, not recommended)
- padded plaintext m:

$$0x00 \,\|\, 0x02 \,\|\, \mathrm{PS} \,\|\, 0x00 \,\|\, m$$

- PS – string of pseudo-random nonzero bytes of length ≥ 8
- various recommendation on using this padding (see RFC 8017)

Encrypting with PKCS#1 v1.5 padding (openssl)

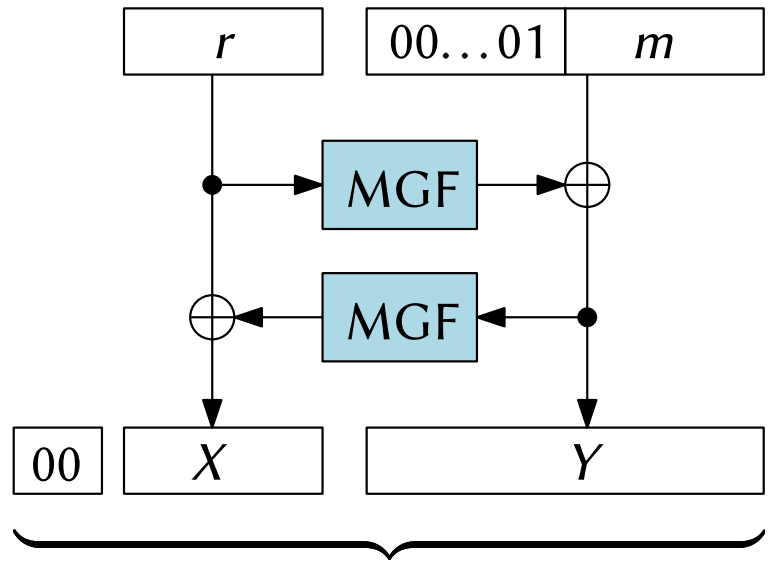
- encryption of short string with RSA using PKCS#1 v1.5 padding (default)
```
echo 'Cryptology - sample plaintext' | openssl
pkeyutl -encrypt -pubin -inkey myrsa-pub.pem -out
cipher.bin
```
- decrypting the ciphertext:
```
openssl pkeyutl -decrypt -inkey myrsa.pem -in
cipher.bin
```
- see the padding:
```
openssl pkeyutl -decrypt -inkey myrsa.pem -in
cipher.bin -hexdump -pkeyopt rsa_padding_mode:none
```

```
0000 - 00 02 e6 88 63 5a d2 e2-57 ee d7 74 6a 78 1a 6a   ....cZ..W..tjx.j
0010 - 75 b2 57 3b 6a d7 bb f8-3b 61 c1 23 bc 77 18 91   u.W;j...;a.#.w..
0020 - bf 2d 36 ed a8 25 30 16-44 d5 ba 88 1e 7c 3e 12   .-6..%0.D....|>.
0030 - d1 c7 b0 7a 3e 5a 47 b3-87 66 7f f0 47 c4 89 c2   ...z>ZG..f..G...
0040 - 69 aa 17 cb e3 96 4f d2-43 8f 4d c5 8c 47 e6 91   i.....O.C.M..G..
0050 - cd e6 0a c0 a3 15 72 9d-0f cb 17 4c 85 43 97 5f   ......r....L.C._
0060 - 52 d1 67 e9 92 8d c5 85-52 ed 15 c7 28 c9 33 b9   R.g.....R...(.3.
0070 - dc 5d 84 b8 6c 26 e3 1a-d7 5a 9a f3 44 cf f1 0d   .]..l&...Z..D...
0080 - 70 b3 3a 22 c1 37 82 e0-7d 24 1d 6d 3e ef e3 b9   p.:".7..}$.m>...
0090 - 3f 2c 56 32 69 27 8a 61-a6 72 3f 98 f3 e7 b2 3a   ?,V2i'.a.r?....:
00a0 - 68 e6 ff c9 cd 90 c6 1b-fc 2a 5e c9 bd cf 38 0c   h........*^...8.
00b0 - 8d 97 08 75 f1 d7 a4 a3-7c 3e 1e 93 31 33 f2 fd   ...u....|>..13..
00c0 - a7 92 f2 1d 61 0d 8c c4-98 df a2 2d ba ca a9 91   ....a......-....
00d0 - 30 f3 a7 89 b5 02 b1 3f-d4 b2 20 e1 b0 b0 6d d0   0......?.. ...m.
00e0 - cd 00 43 72 79 70 74 6f-6c 6f 67 79 20 2d 20 73   ..Cryptology - s
00f0 - 61 6d 70 6c 65 20 70 6c-61 69 6e 74 65 78 74 0a   ample plaintext.
```

OAEP padding

- OAEP – Optimal Asymmetric Encryption Padding
- recommended for new applications (PKCS #1 v2.2, RFC 8017)
- provable secure (in some sense, in some security model)
- slightly simplified OAEP

- 2 round Feistel
- MGF – mask generation function, based on hash function (RO)
- r – random "seed"
- padding verified in decryption
- impossible to create valid ciphertext without encryption



input to RSA encryption

Encrypting with OAEP padding (openssl)

- encrypting a message:
  ```
  echo 'Cryptology - sample plaintext' | openssl
  pkeyutl -encrypt -pubin -inkey myrsa-pub.pem -out
  cipher2.bin -pkeyopt rsa_padding_mode:oaep
  ```
- decrypting the ciphertext:
  ```
  openssl pkeyutl -decrypt -inkey myrsa.pem -in
  cipher2.bin -pkeyopt rsa_padding_mode:oaep
  ```