# Security of RSA

## Cryptology (1)

Martin Stanek

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KI FMFI UK Bratislava

# RSA scheme

- $n = p \cdot q$ (product of two distinct primes)
- $e \cdot d \equiv 1 \pmod{\varphi(n)}$, where $\varphi(n) = (p-1)(q-1)$
- public key: $(e, n)$
- private exponent: $d$
- public/private transforms $E, D : \mathbb{Z}_n \to \mathbb{Z}_n$
    - $E(m) = m^e \bmod n$
    - $D(c) = c^d \bmod n$

# Hybrid encryption

- encrypting long messages
- encryption of message $m$ for recipient $A$ (his public key is $\mathsf{pk}_A$):

$$\langle E_k(m), E^{\mathrm{RSA}}_{\mathsf{pk}_A}(k) \rangle$$

- notation:
  - $E$ – symmetric cipher (e.g. AES)
  - $k$ – random symmetric key for $E$
  - $E^{\mathrm{RSA}}_{\mathsf{pk}_A}$ – RSA encryption with $A$'s public key
- $A$ can decrypt easily
- advantages: key management (asymmetric scheme), speed
- the security depends on both constructions

# Real world – key transport (key encapsulation mechanism)

- usually wrapping symmetric keys, providing confidentiality and integrity
- key transport
  - RFC 5990: Use of the RSA-KEM Key Transport Algorithm in the Cryptographic Message Syntax (CMS)
  - NIST SP 800-56B rev. 2: Recommendation for Pair-Wise Key-Establishment Schemes Using Integer Factorization Cryptography;
    various schemes, e.g., KTS-OAEP: Key-Transport Using RSA-OAEP

## Factorization and RSA

- factorization $\Rightarrow$ compute the private key $\Rightarrow$ decryption (trivial)
- decryption (knowing only the public key) =?$\Rightarrow$ factorization (open)
- knowledge of $\varphi(n)$ is equivalent to factorization
  - [$\Leftarrow$] trivial
  - [$\Rightarrow$] solving 2 equations with 2 variables:

$$n = p \cdot q$$
$$\varphi(n) = (p-1)(q-1)$$

- knowledge of $d$ is equivalent to factorization
  - [$\Leftarrow$] trivial
  - [$\Rightarrow$] more complicated procedure needed
- corollary: do not share $n$ among group of users

**RSA problem**

Given $(e, n)$ and $c \in \mathbb{Z}_n$, compute $m$ such that $m^e \equiv c \pmod{n}$.

- RSA problem is not more difficult than factorization
  - (open problem) Is the RSA problem as difficult as factorization or easier?

# Problems with primes

- specific algorithms for factorization, when $p, q$ satisfy some properties, for example:
  - small $|p - q|$,
  - $p - 1$ (or $q - 1$) without a large prime factor, etc.
- suspicious methods of generating primes, for example:
  - weak or poorly initialized PRNG
  - primes with some internal structure ("optimization")
- Lenstra et al. (2012)
  - 11.4 million RSA moduli (X.509 certificates, PGP keys)
  - 26965 (incl. 10 RSA-2048) vulnerable (shared a single common prime factor)

# Problems with primes (2)

- Bernstein et al. (2013)
  - Taiwan's national *Citizen Digital Certificate* database
  - generated by government-issued smart cards (certified)
  - 3.2 million unique RSA moduli
  - 103 moduli factored by computing the gcd (sharing a prime divisor)
  - observing non-randomness in the primes … 184 distinct 1024-bit RSA keys factored
- Nemec et al. (2017)
  - problem with "FastPrime" method for primes generation implemented in library for particular hardware chips
  - factor public modulus
  - ID cards – Estonia (750.000), Slovakia (300.000), …

- General number field sieve (GNFS)
- heuristic complexity:

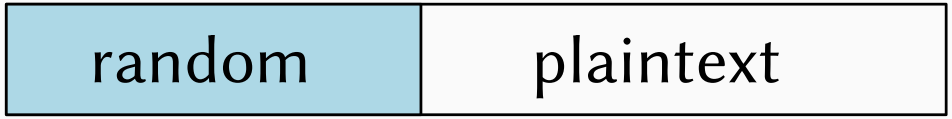
$$\exp\left(\left(\sqrt[3]{64/9}+o(1)\right)(\ln n)^{1/3}(\ln\ln n)^{2/3}\right)$$

- equivalent key lengths
  - NIST Recommendations (SP 800-57 part 1 rev. 5) (2020), see the table ⟶
- other estimates are compared at Keylength.com

| symmetric | RSA |
|---:|---:|
| 80 | 1024 |
| 112 | 2048 |
| 128 | 3072 |
| 192 | 7680 |
| 256 | 15360 |

# Small message (plaintext) space

- RSA scheme is deterministic (the textbook version)
- small plaintext space:
  - such as {yes, no, maybe}
  - attacker can compute $E(m)$ for any $m$ and compare the result with the ciphertext
- potential plaintexts can be tested regardless of plaintext space
- randomization with padding




  - is it secure (can you prove it)?
  - see OAEP for provable security

# Small public exponent – broadcast

- small exponent – speed
- let $e = 3$ for three recipients $A, B, C$ with moduli $n_A, n_B, n_C$
- broadcasting $m$:

$$c_A = m^3 \bmod n_A$$
$$c_B = m^3 \bmod n_B$$
$$c_C = m^3 \bmod n_C$$

- an attacker solves the system of congruences (CRT):

$$x \equiv c_A \pmod{n_A}$$
$$x \equiv c_B \pmod{n_B}$$
$$x \equiv c_C \pmod{n_C}$$

# Small public exponent – broadcast (2)

- solution $x$ (obtained from CRT) and $m^3$ satisfy the system of congruences, thus

$$x \equiv m^3 \pmod{n_A n_B n_C}$$

- $x = m^3$, since $m < n_A, n_B, n_C$
- $m$ can be computed as a cube root of $x$
- padding as a prevention

# Small public exponent – related messages

- $m_1, m_2$ linearly dependent messages; $c_1 = E(m_1), c_2 = E(m_2)$
- $\exists a, b \in \mathbb{Z}$: $m_2 = am_1 + b$, the attacker knows $a, b$
- variable $z$ ($m_1$ is a root of the following polynomials):

$$z^e - c_1 \equiv 0 \pmod{n}$$
$$(az + b)^e - c_2 \equiv 0 \pmod{n}$$

- $(z - m_1)$ divides both polynomials
- $\gcd(z^e - c_1, (az + b)^e - c_2)$ reveals $m_1$, and subsequently $m_2$
- easy to generalize for any known polynomial relation
- prevention: suitable padding
  - not every padding is secure (see Coppersmith's attack)

# Small private exponent

- motivation: fast decryption
- implementation: choose $d$ first, $e$ computed afterward
- results – $d$ can be computed from a public key:
  - Wiener (1990): $d < n^{0.25}/3$ (continued fraction)
  - Boneh, Durfee (1999): $d < n^{0.292}$ (Coppersmith, LLL)
  - some other improvements exist
- do not “optimize” $d$ (!)

# Some applications of Coppersmith's theorem

- Coppersmith's theorem – finding all small solutions of modular polynomial equation
- computing plaintext when using short/improper padding (and small $e$)
- computing primes given some fraction of their bits
- reconstructing $d$ given some fraction of its bits

# Using homomorphism of RSA

- $E(m_1 \cdot m_2) = E(m_1) \cdot E(m_2)$, computations are mod $n$
- let's assume, that $l$-bit symmetric key $k$ is encrypted, i.e. $k < 2^l$
- the attacker pre-computes $E(1), E(2), E(3), \ldots, E\left(2^{l/2}\right)$, and stores the values $\langle E(i), i \rangle$ in a hash table
- if $k = k_1 \cdot k_2$, for $k_i \leq 2^{l/2}$:
  - the attacker tries $k_1 = 1, 2, 3, \ldots, 2^{l/2}$, and searches $c/E(k_1) = E(k/k_1)$ in the table
  - a match yields $k_1, k_2$ (and therefore $k$)
- time complexity $O\left(2^{l/2}\right)$
- increasing the number of pre-computed values ⇒ higher probability of success
- (!) for small $e$, such as $e = 3$, the attacker can compute $\sqrt[3]{c}$ directly (if $k^3 < n$)

# Half predicate

- Knowing a ciphertext – can anything be computed about the plaintext?
- (textbook) RSA is not semantically secure (e.g. testing any plaintext)
- oracle $\text{half}(c) = 0$ if $0 \le m < n/2$, or 1 otherwise
- we decrypt any $c$ using predicate half():

$$\begin{aligned} \text{half}(c) = 0 \quad &\Leftrightarrow \quad m \in \{0, \dots, \lfloor n/2 \rfloor\} \\ \text{half}(c \cdot E(2)) = 0 \quad &\Leftrightarrow \quad m \in \{0, \dots, \lfloor n/4 \rfloor\} \cup \{\lceil 2n/4 \rceil, \dots, \lfloor 3n/4 \rfloor\} \\ \text{half}\left(c \cdot E\left(2^2\right)\right) = 0 \quad &\Leftrightarrow \quad m \in \{0, \dots, \lfloor n/8 \rfloor\} \cup \dots \end{aligned}$$

- we can compute $m$ by binary search; $c \cdot E\left(2^l\right) = E\left(m \cdot 2^l\right)$
- remark: $d$ is not used nor computed in this attack

# Parity predicate

- similarly to half(), we can use the predicate parity()
  - $\text{parity}(c) = m \,\&\, 0\text{x}1$
- relation between predicates: $\text{half}(c) = \text{parity}(c \cdot E(2))$
  - if $0 \leq m < n/2$:
    then $0 \leq 2m < n$ and the plaintext corresponding to $c \cdot E(2)$ is even
  - if $n/2 < m < n$:
    then $n \leq 2m < 2n \quad \Rightarrow \quad 2m \bmod n = 2m - n$,
    i.e. the plaintext corresponding to $c \cdot E(2)$ is odd

# Bleichenbacher's attack on PKCS #1 v1.5 (1)

- chosen ciphertext attack (1998)
- PKCS#1 v1.5 oracle (error message, timing, etc.) ⇒ decryption of arbitrary ciphertext
- PKCS#1 v1.5 padding:

| | | | | |
|---|---|---|---|---|
| `00` | `02` | $\geq 8$ non-zero bytes | `00` | message |

- $k$ – byte length of $n$; $2^{8(k-1)} \leq n < 2^{8k}$
- PKCS conforming block:
  1. starts with bytes `00 02`
  2. bytes 3, ..., 10 are non-zero
  3. there is some `00` byte later (bytes 11, ..., $k$)
- let's denote $B = 2^{8(k-2)}$, i.e. PKCS conforming block: $2B \leq m < 3B$
- ciphertext is called PKCS conforming if its decryption is PKCS conf.

# Bleichenbacher's attack on PKCS #1 v1.5 (2)

- given $c \in \mathbb{Z}_n$ the attacker wants to compute $m = c^d \bmod n$
- modifying $c$ and testing PKCS conformity
- sequence of gradually narrower intervals for $m$
- single element $m$ at the end

# Bleichenbacher's attack on PKCS #1 v1.5 (3)

- Impact:
  - SSL/TLS ($\leq$ 1.2) RSA key exchange method: client sends *pre-master secre* encrypted with server's public key (PKCS#1 v1.5)
  - decryption of the pre-master secret yields the session keys
  - careful implementation needed, see TLS 1.2 (RFC 5246)
  - when relevant, the attack allows to create a PKCS #1 v1.5 signature of arbitrary message (using server's private key)
- ROBOT (Return Of Bleichenbacher's Oracle Threat)
  - attack on TLS after 19 years (2018)
  - advice: disable all TLS_RSA ciphersuits
  - non-standard message flow (shortened)
  - different responses: different alert codes, TCP FIN, TCP timeout, TCP reset, two alerts …

# Manger’s attack

- Does OAEP help (it is almost impossible to generate a valid ciphertext)?
- Manger’s attack (2001): compute $m = c^d \bmod n$ for any $c$
- assumption: access to the following oracle:
  - Given $c'$, is the first byte of $(c')^d \bmod n$ zero?
  - let $k$ be the byte length of $n$, and $B = 2^{8(k-1)}$
  - oracle: $(c')^d \bmod n < B$
- recognizing bad first byte vs. bad internal integrity of decrypted block
- gradually reduce an interval of possible $m$ values
- can be adapted to PKCS #1 v1.5
- there are also improvements to Bleichenbacher’s attack

- Timing attacks
  - straightforward implementation of modular exponentiation
  - computation time of $D(c)$ depends on $c$, $d$, and $n$
  - statistical correlation analysis to recover $d$ from many samples $(c_i, \text{time}_i)$
  - variant used to attack SSL implementation (2003) with approx. million queries for extracting private key and factoring 1024 bit modulus $n$
  - prevention: blinding
- Fault attacks
  - induce faults while executing sensitive operations
  - heat, power spikes, clock glitches, etc.
  - example: fault in a single value/computation in RSA CRT (signature computation) – correct and fault signatures yield the factorization of $n$

# Exercises

1. *RSA scheme with* $e = 65537$ *and* $n =$

   8018608475076248861401338396528153578287820945201836824795780800244524288592277269106414230276039139835483315126023263502834747291110297421450739193636422493636284898957959717534955334889355392634316873456715784103318815035259216630761248292597761841915997817969297254535111670315043397860030969596000797926038535677581878499822107307972159780436264034774632677

   *Knowing that the primes are close to each other, decrypt this ciphertext:*

   3452729114333932334376815266108276901304755982160575827634415443749519749211808268506238634948363286684638899883895161032893436197627294304033289198776700586095373121976624551002462095586197518311489710032519588687787308657444270122109211925263816514424667137213129991316629085928018258448714415908698927481775112071167884592192687879962398266421762231336132205

2. *Create and test an example for decrypting linearly dependent plaintext encrypted without padding in RSA scheme with small public exponent.*